

SECTION – A

1. Consider three observations a , b and c such that $b = a+c$. If the standard deviation of $a+2$, $b+2$, $c+2$ is d , then which of the following is true?

(1) $b^2 = a^2 + c^2 + 3d^2$

(2) $b^2 = 3(a^2 + c^2) - 9d^2$

(3) $b^2 = 3(a^2 + c^2) + 9d^2$

(4) $b^2 = 3(a^2 + c^2 + d^2)$

Ans. (2)

Sol. for a , b , c

$$\text{mean} = \bar{x} = \frac{a+b+c}{3}$$

$$\bar{x} = \frac{2b}{3}$$

S.D. of a , b , $c = d$

$$d^2 = \frac{a^2 + b^2 + c^2}{3} - \frac{4b^2}{9}$$

$$b^2 = 3a^2 + 3c^2 - 9d^2$$

2. Let a vector $\alpha\hat{i} + \beta\hat{j}$ be obtained by rotating the vector $\sqrt{3}\hat{i} + \hat{j}$ by an angle 45° about the origin in counter clockwise direction in the first quadrant. Then the area of triangle having vertices (α, β) , $(0, \beta)$ and $(0, 0)$ is equal to :

(1) 1

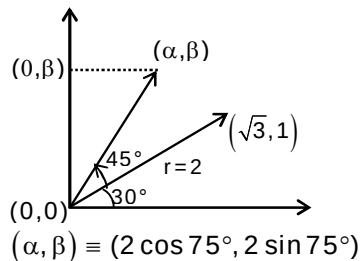
(2) $\frac{1}{2}$

(3) $\frac{1}{\sqrt{2}}$

(4) $2\sqrt{2}$

Ans. (2)

Sol.



$$\text{Area} = \frac{1}{2} (2 \cos 75^\circ)(2 \sin 75^\circ)$$

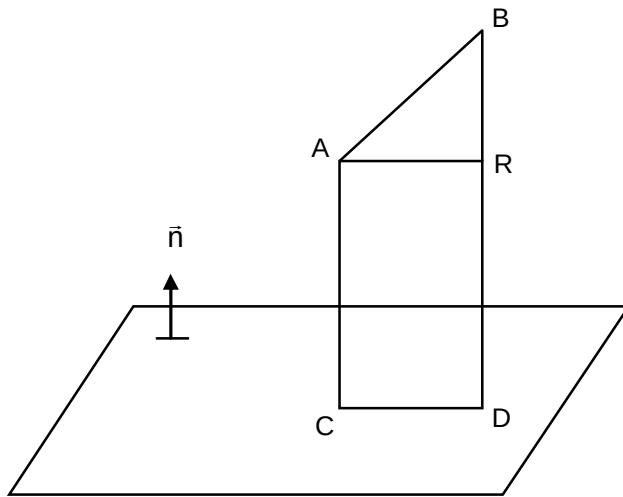
$$= \sin(150^\circ) = \frac{1}{2} \text{ square unit}$$

3. If for $a > 0$, the feet of perpendiculars from the points $A(a, -2a, 3)$ and $B(0, 4, 5)$ on the plane $lx + my + nz = 0$ are points $C(0, -a, -1)$ and D respectively, then the length of line segment CD is equal to :

- (1) $\sqrt{41}$
 (2) $\sqrt{55}$
 (3) $\sqrt{31}$
 (4) $\sqrt{66}$

Ans. (4)

Sol.



$$CD = AR = |AB| \sin \phi$$

$$CD = |AB| \sqrt{1 - \cos^2 \phi}$$

$$|AB| \sqrt{1 - \left(\frac{\vec{AB} \cdot \vec{n}}{|\vec{AB}| |\vec{n}|} \right)^2}$$

$$= \sqrt{(AB)^2 - (\vec{AB} \cdot \vec{n})^2}$$

$$\cos \phi = \frac{\vec{AB} \cdot \vec{n}}{|\vec{n}| |\vec{AB}|}$$

$$|\vec{AB}| = a\hat{i} - (2a+4)\hat{j} - 2\hat{k}$$

$$\vec{AB} \cdot \vec{n} = la - (2a+4)m - 2n$$

C on plane

$$0l - am - n = 0 \quad \dots (1)$$

$$\vec{AC} \parallel \vec{n}$$

$$\frac{a}{\ell} = \frac{-a}{m} = \frac{4}{n}$$

$$m = -\ell \text{ \& \; } an + 4m = 0 \quad \dots (2)$$

From (1) and (2)

$$a^2m + an = 0$$

$$\underline{4m + an = 0}$$

$$(a^2 - 4)m = 0 \Rightarrow \boxed{a = 2}.$$

$$2m + n = 0 \quad \dots (1)$$

$$m + \ell = 0$$

$$\ell^2 + m^2 + n^2 = 1$$

$$m^2 + m^2 + 4m^2 = 1$$

$$m^2 = \frac{1}{6}$$

$$m = \frac{1}{\sqrt{6}}$$

$$n = \frac{-2}{\sqrt{6}}$$

$$\ell = \frac{-1}{\sqrt{6}}$$

$$\text{Now } \overrightarrow{AB} \cdot \vec{n} = 2 \left(\frac{-1}{\sqrt{6}} \right) - 8 \left(\frac{1}{\sqrt{6}} \right) - 2 \left(\frac{-2}{\sqrt{6}} \right)$$

$$= \frac{-2-8+4}{\sqrt{6}} = -\sqrt{6}$$

$$|\overrightarrow{AB}| = \sqrt{4+64+4} = \sqrt{72}$$

$$CD = \sqrt{72-6}$$

$$CD = \sqrt{66}$$

4. The range of $a \in \mathbb{R}$ for which the function

$f(x) = (4a-3)(x + \log_e 5) + 2(a-7) \cot\left(\frac{x}{2}\right) \sin^2\left(\frac{x}{2}\right)$, $x \neq 2n\pi, n \in \mathbb{N}$ has critical points, is :

$$(1) \left[-\frac{4}{3}, 2 \right]$$

$$(2) [1, \infty)$$

$$(3) (-\infty, -1]$$

$$(4) (-3, 1)$$

Ans. (1)

Sol. $f(x) = (4a - 3)(x + \ln 5) + 2(a - 7) \left(\frac{\cos \frac{x}{2}}{\sin \frac{x}{2}} \cdot \sin^2 \frac{x}{2} \right)$

$$f(x) = (4a - 3)(x + \ln 5) + (a - 7) \sin x$$

$$f'(x) = (4a - 3) + (a - 7) \cos x = 0$$

$$\cos x = \frac{-(4a - 3)}{a - 7}$$

$$-1 \leq -\frac{4a - 3}{a - 7} \leq 1$$

$$-1 \leq \frac{4a - 3}{a - 7} \leq 1$$

$$\frac{4a - 3}{a - 7} - 1 \leq 0 \text{ and } \frac{4a - 3}{a - 7} + 1 \geq 0$$

$$\Rightarrow \frac{-4}{3} \leq a \leq 2$$

5. Let the functions $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be defined as :

$$f(x) = \begin{cases} x + 2, & x < 0 \\ x^2, & x \geq 0 \end{cases} \text{ and } g(x) = \begin{cases} x^3, & x < 1 \\ 3x - 2, & x \geq 1 \end{cases}$$

Then, the number of points in $\mathbb{R}$ where $(f \circ g)(x)$ is NOT differentiable is equal to :

$$(1) 1$$

$$(2) 2$$

$$(3) 3$$

$$(4) 0$$

Ans. (1)

Sol. $f \circ g(x) = \begin{cases} x^3 + 2, & x \leq 0 \\ x^6, & 0 \leq x \leq 1 \\ (3x - 2)^2, & x \geq 1 \end{cases}$

$\therefore f \circ g(x)$ is discontinuous at $x = 0$ then non-differentiable at $x = 0$

Now,

at $x = 1$

$$\text{RHD} = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{(3(1+h) - 2)^2 - 1}{h} = 6$$

$$\text{LHD} = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \rightarrow 0} \frac{(1-h)^6 - 1}{-h} = 6$$

Number of points of non-differentiability = 1

6. Let a complex number z , $|z| \neq 1$, satisfy $\log_{\frac{1}{\sqrt{2}}} \left(\frac{|z|+11}{(|z|-1)^2} \right) \leq 2$. Then, the largest value of $|z|$ is

equal to _____

- (1) 5
- (2) 8
- (3) 6
- (4) 7

Ans. (4)

Sol. $\frac{|z|+11}{(|z|-1)^2} \geq \frac{1}{2}$

$$\begin{aligned} 2|z| + 22 &\geq (|z| - 1)^2 \\ 2|z| + 22 &\geq |z|^2 - 2|z| + 1 \\ |z|^2 - 4|z| - 21 &\leq 0 \\ (|z| - 7)(|z| + 3) &\leq 0 \\ \Rightarrow |z| &\leq 7 \\ \therefore |z|_{\max} &= 7 \end{aligned}$$

7. A pack of cards has one card missing. Two cards are drawn randomly and are found to be spades. The probability that the missing card is not a spade, is :

- (1) $\frac{3}{4}$
- (2) $\frac{52}{867}$
- (3) $\frac{39}{50}$
- (4) $\frac{22}{425}$

Ans. (3)

Sol. $P(\overline{S_{\text{missing}}} / \text{both found spade}) = \frac{P(\overline{S_m} \cap \text{BFS})}{P(\text{BFS})}$

$$\begin{aligned} &= \frac{\left(1 - \frac{13}{52}\right) \times \frac{13}{51} \times \frac{12}{50}}{\left(1 - \frac{13}{52}\right) \times \frac{13}{51} \times \frac{12}{50} + \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50}} \\ &= \frac{39}{50} \end{aligned}$$

8. If n is the number of irrational terms in the expansion of $\left(3^{\frac{1}{4}} + 5^{\frac{1}{8}}\right)^{60}$, then $(n-1)$ is divisible by :

- (1) 8
- (2) 26
- (3) 7
- (4) 30

Ans. (2)

Sol. $T_{r+1} = {}^{60}C_r (3^{1/4})^{60-r} (5^{1/8})^r$

rational if $\frac{60-r}{4}, \frac{r}{8}$, both are whole numbers, $r \in \{0,1,2,\dots,60\}$

$$\frac{60-r}{4} \in W \Rightarrow r \in \{0,4,8,\dots,60\}$$

$$\text{and } \frac{r}{8} \in W \Rightarrow r \in \{0,8,16,\dots,56\}$$

$\therefore$ Common terms $r \in \{0,8,16,\dots,56\}$

So 8 terms are rational

Then Irrational terms $= 61 - 8 = 53 = n$

$$\therefore n - 1 = 52 = 13 \times 2^2$$

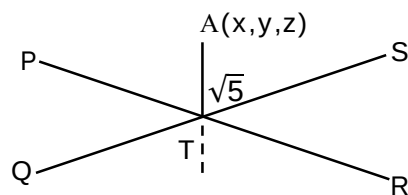
factors 1,2,4,13,26,52

9. Let the position vectors of two points P and Q be $3\hat{i} - \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} - 4\hat{k}$, respectively. Let R and S be two points such that the direction ratios of lines PR and QS are $(4, -1, 2)$ and $(-2, 1, -2)$ respectively. Let lines PR and QS intersect at T. If the vector $\vec{TA}$ is perpendicular to both $\vec{PR}$ and $\vec{QS}$ and the length of vector $\vec{TA}$ is $\sqrt{5}$ units, then the modulus of a position vector of A is :

- (1) $\sqrt{5}$
 (2) $\sqrt{171}$
 (3) $\sqrt{227}$
 (4) $\sqrt{482}$

Ans. (2)

Sol. $\vec{p} = 3\hat{i} - \hat{j} + 2\hat{k}$ & $\vec{q} = \hat{i} + 2\hat{j} - 4\hat{k}$
 $\vec{v}_{PR} = \langle 4, -1, 2 \rangle$ & $\vec{v}_{QS} = \langle -2, 1, -2 \rangle$



$$L_{PR} : \vec{r} = (3\hat{i} - \hat{j} + 2\hat{k}) + \lambda \langle 4, -1, 2 \rangle$$

$$L_{QS} : \vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \mu \langle -2, 1, -2 \rangle$$

$$\text{Now T on PR} = \langle 3 + 4\lambda, -1 - \lambda, 2 + 2\lambda \rangle$$

$$\text{Similarly T on QS} = \langle 1 - 2\mu, 2 + \mu, -4 - 2\mu \rangle$$

$$\text{For } \lambda \text{ \& \; } \mu : \begin{cases} 3 + 4\lambda = 1 - 2\mu \Rightarrow \mu + 2\lambda = -1 \\ -1 - \lambda = 2 + \mu \Rightarrow \mu + \lambda = -3 \end{cases} \quad \begin{matrix} \lambda = 2 \\ \mu = -5 \end{matrix}$$

$$2 + 2\lambda = -4 - 2\mu$$

$$\Rightarrow T : \langle 11, -3, 6 \rangle$$

$$\text{D.R. of TA} = \vec{v}_{QS} \times \vec{v}_{PR}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -2 \\ 4 & -1 & 2 \end{vmatrix} = 0\hat{i} - 4\hat{j} - 2\hat{k}$$

$$L_{TA} : \vec{r} = (11\hat{i} - 3\hat{j} + 6\hat{k}) + \lambda \langle -4\hat{j} - 2\hat{k} \rangle$$

$$\text{Now } A = \langle 11, -3 - 4\lambda, 6 - 2\lambda \rangle$$

$$TA = \sqrt{5}$$

$$(-3 + 4\lambda + 3)^2 + (6 + 2\lambda - 6)^2 = 5$$

$$16\lambda^2 + 4\lambda^2 = 5$$

$$20\lambda^2 = 5$$

$$\lambda = \pm \frac{1}{2}$$

$$A: (11, -3 - 2, 6 - 1) ; \quad A: (11, -3 + 2, 6 + 1)$$

$$|A| = \sqrt{121 + 25 + 25} ; \quad |A| = \sqrt{121 + 1 + 49}$$

$$= \sqrt{171} ; \quad \sqrt{171}$$

- 10.** If the three normals drawn to the parabola, $y^2 = 2x$ pass through the point $(a, 0)$ $a \neq 0$, then 'a' must be greater than:

(1) 1

(2) $\frac{1}{2}$

(3) $-\frac{1}{2}$

(4) -1

Ans. (1)

Sol. Let the equation of the normal is

$$y = mx - 2am - am^3$$

$$\text{here } 4a = 2 \Rightarrow a = \frac{1}{2}$$

$$y = mx - m - \frac{1}{2}m^3$$

It passing through $A(a, 0)$ then

$$0 = am - m - \frac{1}{2}m^3$$

$$m = 0, a - 1 - \frac{1}{2}m^2 = 0$$

$$m^2 = 2(a - 1) > 0$$

$$\therefore a > 1$$

- 11.** Let $S_k = \sum_{r=1}^k \tan^{-1} \left(\frac{6^r}{2^{2r+1} + 3^{2r+1}} \right)$. Then $\lim_{k \rightarrow \infty} S_k$ is equal to :

(1) $\tan^{-1}\left(\frac{3}{2}\right)$

(2) $\cot^{-1}\left(\frac{3}{2}\right)$

(3) $\frac{\pi}{2}$

(4) $\tan^{-1}(3)$

Ans. (2)

Sol.
$$\sum_{r=1}^{\infty} \tan^{-1} \left(\frac{6^r (3-2)}{\left(1 + \left(\frac{3}{2}\right)^{2r+1}\right) 2^{2r+1}} \right)$$

$$\sum_{r=1}^{\infty} \tan^{-1} \left(\frac{2^r \cdot 3^{r+1} - 3^r 2^{r+1}}{\left(1 + \left(\frac{3}{2}\right)^{2r+1}\right) 2^{2r+1}} \right)$$

$$\sum_{r=1}^{\infty} \tan^{-1} \left(\frac{\left(\frac{3}{2}\right)^{r+1} - \left(\frac{3}{2}\right)^r}{1 + \left(\frac{3}{2}\right)^{r+1} \left(\frac{3}{2}\right)^r} \right) = \sum_{r=1}^{\infty} \left[\tan^{-1} \left(\frac{3}{2}\right)^{r+1} - \tan^{-1} \left(\frac{3}{2}\right)^r \right] = \frac{\pi}{2} - \tan^{-1} \frac{3}{2} = \cot^{-1} \frac{3}{2}$$

12. The number of roots of the equation, $(81)^{\sin^2 x} + (81)^{\cos^2 x} = 30$ in the interval $[0, \pi]$ is equal to :

(1) 3

(2) 2

(3) 4

(4) 8

Ans. (3)

Sol. $(81)^{\sin^2 x} + (81)^{1-\sin^2 x} = 30$

$$(81)^{\sin^2 x} + \frac{81}{(81)^{\sin^2 x}} = 30$$

Let $(81)^{\sin^2 x} = t$

$$t + \frac{81}{t} = 30 \Rightarrow t^2 + 81 = 30t$$

$$t^2 - 30t + 81 = 0$$

$$t^2 - 27t - 3t + 81 = 0$$

$$(t - 3)(t - 27) = 0$$

$$t = 3, 27$$

$$(81)^{\sin^2 x} = 3, 3^3$$

$$3^{4 \sin^2 x} = 3^1, 3^3$$

$$4 \sin^2 x = 1, 3$$

$$\sin^2 x = \frac{1}{4}, \frac{3}{4}$$

$$\text{in } [0, \pi] \sin x > 0$$

$$\sin x = \frac{1}{2}, \frac{\sqrt{3}}{2}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}$$

$$\text{Number of solution} = 4$$

- 13.** If $y=y(x)$ is the solution of the differential equation, $\frac{dy}{dx} + 2y \tan x = \sin x$, $y\left(\frac{\pi}{3}\right) = 0$, then the maximum value of the function $y(x)$ over R is equal to :

(1) 8

(2) $\frac{1}{2}$

(3) $-\frac{15}{4}$

(4) $\frac{1}{8}$

Ans. (4)

Sol. $\frac{dy}{dx} + 2 \tan x \cdot y = \sin x$

$$\text{I.F.} = e^{2 \int \tan x} = \sec^2 x$$

$$y \cdot \sec^2 x = \int \sin x \sec^2 x dx = \int \tan x \sec x dx + c$$

$$y \sec^2 x = \sec x + c$$

$$y = \cos x + c \cos^2 x$$

$$x = \frac{\pi}{3}, y = 0$$

$$\Rightarrow \frac{1}{2} + \frac{c}{4} \Rightarrow c = -2$$

$$\therefore y = \cos x - 2 \cos^2 x$$

$$y = -2 \left(\cos^2 x - \frac{1}{2} \cos x \right) = -2 \left(\left(\cos x - \frac{1}{4} \right)^2 - \frac{1}{16} \right)$$

$$y = \frac{1}{8} - 2 \left(\cos x - \frac{1}{4} \right)^2$$

$$\therefore y_{\max} = \frac{1}{8}$$

- 14.** Which of the following Boolean expression is a tautology?

- (1) $(p \wedge q) \wedge (p \rightarrow q)$
 (2) $(p \wedge q) \vee (p \vee q)$
 (3) $(p \wedge q) \vee (p \rightarrow q)$
 (4) $(p \wedge q) \rightarrow (p \rightarrow q)$

Ans. (4)

Sol.	p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$(p \wedge q) \rightarrow (p \rightarrow q)$
	T	T	T	T	T	T
	F	T	F	T	T	T
	T	F	F	T	F	T
	F	F	F	F	T	T

15. Let $A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$, $i = \sqrt{-1}$. Then, the system of linear equations $A^8 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$ has :

- (1) No solution
 (2) Exactly two solutions
 (3) A unique solution
 (4) Infinitely many solutions

Ans. (1)

Sol. $A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$

$$A^2 = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix} \begin{bmatrix} i & -i \\ -i & i \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} = 2 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$A^4 = 4 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = 4 \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 8 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A^8 = 64 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 64 \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 128 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$128 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$$

$$128 \begin{bmatrix} x - y \\ -x + y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix} \Rightarrow 128(x - y) = 8$$

$$\Rightarrow x - y = \frac{1}{16} \dots (1)$$

and $128(-x + y) = 64 \Rightarrow x - y = \frac{-1}{2} \dots (2)$

$\Rightarrow$ no solution (from eq. (1) & (2))

16. If for $x \in \left(0, \frac{\pi}{2}\right)$, $\log_{10} \sin x + \log_{10} \cos x = -1$ and $\log_{10}(\sin x + \cos x) = \frac{1}{2} (\log_{10} n - 1)$, $n > 0$, then the value of n is equal to :

(1) 16
 (2) 20
 (3) 12

(4) 9

Ans. (3)

Sol. $\log_{10} (\sin x) + \log_{10} (\cos x) = -1$

$$\sin x \cos x = \frac{1}{10} \dots (1)$$

$$\text{and } \log_{10} (\sin x + \cos x) = \frac{1}{2} (\log_{10} n - 1)$$

$$\Rightarrow \sin x + \cos x = \left(\frac{n}{10} \right)^{\frac{1}{2}}$$

$$\Rightarrow \sin^2 x + \cos^2 x + 2 \sin x \cos x = \frac{n}{10} \text{ (squaring)}$$

$$\Rightarrow 1 + 2 \left(\frac{1}{10} \right) = \frac{n}{10} \text{ (using equation (1))}$$

$$\Rightarrow \frac{n}{10} = \frac{12}{10} \Rightarrow n = 12$$

- 17.** The locus of the midpoints of the chord of the circle, $x^2 + y^2 = 25$ which is tangent to the hyperbola, $\frac{x^2}{9} - \frac{y^2}{16} = 1$ is :

$$(1) (x^2 + y^2)^2 - 16x^2 + 9y^2 = 0$$

$$(2) (x^2 + y^2)^2 - 9x^2 + 144y^2 = 0$$

$$(3) (x^2 + y^2)^2 - 9x^2 - 16y^2 = 0$$

$$(4) (x^2 + y^2)^2 - 9x^2 + 16y^2 = 0$$

Ans. (4)

Sol. tangent of hyperbola

$$y = mx \pm \sqrt{9m^2 - 16} \dots (i)$$

which is a chord of circle with mid-point (h, k)

so equation of chord T = S₁

$$hx + ky = h^2 + k^2$$

$$y = -\frac{hx}{k} + \frac{h^2 + k^2}{k} \dots (ii)$$

by (i) and (ii)

$$m = -\frac{h}{k} \text{ and } \sqrt{9m^2 - 16} = \frac{h^2 + k^2}{k}$$

$$9 \frac{h^2}{k^2} - 16 = \frac{(h^2 + k^2)^2}{k^2}$$

$$\text{locus } 9x^2 - 16y^2 = (x^2 + y^2)^2$$

18. Let $[x]$ denote greatest integer less than or equal to x . If for $n \in \mathbb{N}$, $(1 - x + x^3)^n = \sum_{j=0}^{3n} a_j x^j$, then

$$\sum_{j=0}^{\left[\frac{3n}{2}\right]} a_{2j} + 4 \sum_{j=0}^{\left[\frac{3n-1}{2}\right]} a_{2j+1} \text{ is equal to :}$$

- (1) 1
(2) n
(3) 2^{n-1}
(4) 2

Ans. (1)

Sol. $(1 - x + x^3)^n = \sum_{j=0}^{3n} a_j x^j$

$$(1 - x + x^3)^n = a_0 + a_1 x + a_2 x^2 + \dots + a_{3n} x^{3n}$$

Put $x = 1$

$$1 = a_0 + a_1 + a_2 + a_3 + a_4 + \dots + a_{3n} \quad \dots(1)$$

Put $x = -1$

$$1 = a_0 - a_1 + a_2 - a_3 + a_4 - \dots - (-1)^{3n} a_{3n} \quad \dots(2)$$

Add (1) + (2)

$$\Rightarrow a_0 + a_2 + a_4 + a_6 + \dots = 1$$

Sub (1) - (2)

$$\Rightarrow a_1 + a_3 + a_5 + a_7 + \dots = 0$$

$$\text{Now } \sum_{j=0}^{\left[\frac{3n}{2}\right]} a_{2j} + 4 \sum_{j=0}^{\left[\frac{3n-1}{2}\right]} a_{2j+1}$$

$$= (a_0 + a_2 + a_4 + \dots) + 4(a_1 + a_3 + \dots)$$

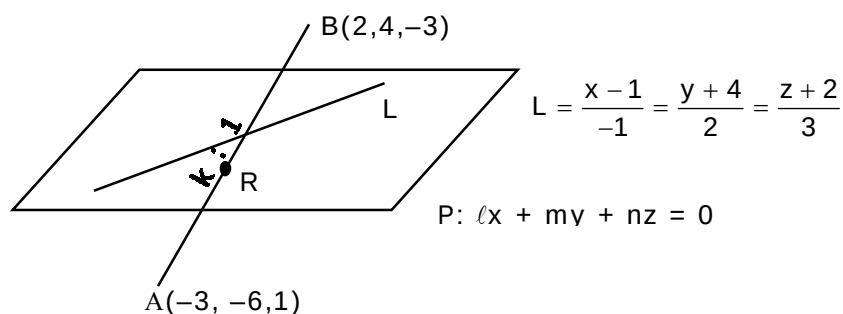
$$= 1 + 4 \times 0$$

$$= 1$$

19. Let P be a plane $lx + my + nz = 0$ containing the line, $\frac{1-x}{1} = \frac{y+4}{2} = \frac{z+2}{3}$. If plane P divides the line segment AB joining points $A(-3, -6, 1)$ and $B(2, 4, -3)$ in ratio $k : 1$ then the value of k is equal to :

- (1) 1.5
(2) 2
(3) 4
(4) 3

Ans. (2)



Sol.

Line lies on plane

$$-\ell + 2m + 3n = 0 \quad \dots(1)$$

Point on line $(1, -4, -2)$ lies on plane

$$\ell - 4m - 2n = 0 \quad \dots(2)$$

from (1) & (2)

$$-2m + n = 0 \Rightarrow 2m = n$$

$$\ell = 3n + 2m \Rightarrow \ell = 4n$$

$$\ell : m : n :: 4n : \frac{n}{2} : n$$

$$\ell : m : n :: 8n : n : 2n$$

$$\ell : m : n :: 8 : 1 : 2$$

Now equation of plane is $8x + y + 2z = 0$

R divide AB in ratio $k : 1$

$$R : \left(\frac{-3+2k}{k+1}, \frac{-6+4k}{k+1}, \frac{1-3k}{k+1} \right) \text{ lies on plane}$$

$$8 \left(\frac{-3+2k}{k+1} \right) + \left(\frac{-6+4k}{k+1} \right) + 2 \left(\frac{1-3k}{k+1} \right) = 0$$

$$-24 + 16k - 6 + 4k + 2 - 6k = 0$$

$$-28 + 14k = 0$$

$$k = 2$$

20. The number of elements in the set $\{x \in \mathbb{R} : (|x| - 3) |x + 4| = 6\}$ is equal to :

(1) 2

(2) 1

(3) 3

(4) 4

Ans. (1)

Sol. Case-1 $x \leq -4$

$$(-x - 3)(-x - 4) = 6$$

$$\Rightarrow (x + 3)(x + 4) = 6$$

$$\Rightarrow x^2 + 7x + 6 = 0$$

$$\Rightarrow x = -1 \text{ or } -6$$

$$\text{but } x \leq -4$$

$$x = -6$$

$$\textbf{Case-2 } x \in (-4, 0)$$

$$(-x - 3)(x + 4) = 6$$

$$\Rightarrow -x^2 - 7x - 12 - 6 = 0$$

$$\Rightarrow x^2 + 7x + 18 = 0$$

$$D < 0 \text{ No solution}$$

$$\textbf{Case-3 } x \geq 0$$

$$(x - 3)(x + 4) = 6$$

$$\Rightarrow x^2 + x - 12 - 6 = 0$$

$$\Rightarrow x^2 + x - 18 = 0$$

$$x = \frac{-1 \pm \sqrt{1 + 72}}{2}$$

$$\therefore x = \frac{\sqrt{73} - 1}{2} \text{ only}$$

SECTION -B

- 1.** Let $f: (0, 2) \rightarrow \mathbb{R}$ be defined as $f(x) = \log_2 \left(1 + \tan \left(\frac{\pi x}{4} \right) \right)$. Then, $\lim_{n \rightarrow \infty} \frac{2}{n} \left(f \left(\frac{1}{n} \right) + f \left(\frac{2}{n} \right) + \dots + f(1) \right)$ is equal to _____

Ans. (1)

Sol. $E = 2 \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} f \left(\frac{r}{n} \right)$

$$E = \frac{2}{\ln 2} \int_0^1 \ln \left(1 + \tan \frac{\pi x}{4} \right) dx \quad \dots (i)$$

replacing $x \rightarrow 1 - x$

$$E = \frac{2}{\ln 2} \int_0^1 \ln \left(1 + \tan \frac{\pi}{4} (1 - x) \right) dx$$

$$E = \frac{2}{\ln 2} \int_0^1 \ln \left(1 + \tan \left(\frac{\pi}{4} - \frac{\pi}{4} x \right) \right) dx$$

$$E = \frac{2}{\ln 2} \int_0^1 \ln \left(1 + \frac{1 - \tan \frac{\pi}{4} x}{1 + \tan \frac{\pi}{4} x} \right) dx$$

$$E = \frac{2}{\ln 2} \int_0^1 \ln \left(\frac{2}{1 + \tan \frac{\pi x}{4}} \right) dx$$

$$E = \frac{2}{\ln 2} \int_0^1 \left(\ln 2 - \ln \left(1 + \tan \frac{\pi x}{4} \right) \right) dx \dots (ii)$$

equation (i) + (ii)

$$E = 1$$

2. The total number of 3×3 matrices A having entries from the set $\{0, 1, 2, 3\}$ such that the sum of all the diagonal entries of AA^T is 9, is equal to _____

Ans. (766)

Sol. $AA^T = \begin{bmatrix} x & y & z \\ a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} x & a & d \\ y & b & e \\ z & c & f \end{bmatrix}$

$$= \begin{bmatrix} x^2 + y^2 + z^2 & ax + by + cz & dx + ey + fz \\ ax + by + cz & a^2 + b^2 + c^2 & ad + be + cf \\ dx + ey + fz & ad + be + cf & d^2 + e^2 + f^2 \end{bmatrix}$$

$$\text{Tr}(AA^T) = x^2 + y^2 + z^2 + a^2 + b^2 + c^2 + d^2 + e^2 + f^2 = 9$$

all $\rightarrow 1$

1

one 3, rest = 0

$$\frac{9!}{8!} = 9$$

two 2, one 1 & rest 0

$$\frac{9!}{2!6!} = 63 \times 4 = 252$$

one 2, five 1, rest 0

$$\frac{9!}{5!3!} = 63 \times 8 = 504$$

$$= 766$$

3. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f(x) + f(x+1) = 2$, for all $x \in \mathbb{R}$. If $I_1 = \int_0^8 f(x) dx$ and

$$I_2 = \int_{-1}^3 f(x) dx, \text{ then the value of } I_1 + 2I_2 \text{ is equal to } \underline{\hspace{2cm}}$$

Ans. (16)

Sol. $f(x) + f(x+1) = 2 \dots (i)$

$$x \rightarrow (x+1)$$

$$f(x+1) + f(x+2) = 2 \dots (ii)$$

by (i) & (ii)

$$f(x) - f(x+2) = 0$$

$$f(x+2) = f(x)$$

$f(x)$ is periodic with $T = 2$

$$I_1 = \int_0^{2 \times 4} f(x) dx = 4 \int_0^2 f(x) dx$$

$$I_2 = \int_{-1}^3 f(x) dx = \int_0^4 f(x+1) dx = \int_0^4 (2 - f(x)) dx$$

$$I_2 = 8 - 2 \int_0^2 f(x) dx$$

$$I_1 + 2I_2 = 16$$

4. Consider an arithmetic series and a geometric series having four initial terms from the set $\{11, 8, 21, 16, 26, 32, 4\}$. If the last terms of these series are the maximum possible four digit numbers, then the number of common terms in these two series is equal to _____

Ans. (3)

Sol. AP – 11, 16, 21, 26

GP – 4, 8, 16, 32

So common terms are 16, 256, 4096

5. If the normal to the curve $y(x) = \int_0^x (2t^2 - 15t + 10) dt$ at a point (a, b) is parallel to the line $x + 3y = -5$, $a > 1$, then the value of $|a + 6b|$ is equal to _____

Ans. (406)

Sol. $y'(x) = (2x^2 - 15x + 10)$

at point P

$$3 = (2a^2 - 15a + 10)$$

$$\Rightarrow 2a^2 - 15a + 7 = 0$$

$$\Rightarrow 2a^2 - 14a - a + 7 = 0$$

$$\Rightarrow 2a(a - 7) - 1(a - 7) = 0$$

$$a = \frac{1}{2} \text{ or } 7,$$

given $a > 1 \therefore a = 7$

also P lies on curve

$$\therefore b = \int_0^a (2t^2 - 15t + 10) dt$$

$$b = \int_0^7 (2t^2 - 15t + 10) dt$$

$$6b = -413$$

$$\therefore |a + 6b| = 406$$

6. If $\lim_{x \rightarrow 0} \frac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2$, then $a + b + c$ is equal to _____

Ans. (4)

$$\text{Sol. } \lim_{x \rightarrow 0} \frac{\left\{ a \left(1 + x + \frac{x^2}{2!} + \dots \right) - b \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots \right) + c \left(1 - x + \frac{x^2}{2!} \dots \right) \right\}}{x \left(x - \frac{x^3}{3!} + \dots \right)} = 2$$

$$\therefore \lim_{x \rightarrow 0} \frac{(a - b + c) + x(a - c) + x^2 \left(\frac{a}{2} + \frac{b}{2} + \frac{c}{2} \right) + \dots}{x^2 \left(1 - \frac{x^2}{6} \dots \right)} = 2$$

$$\therefore a - b + c = 0$$

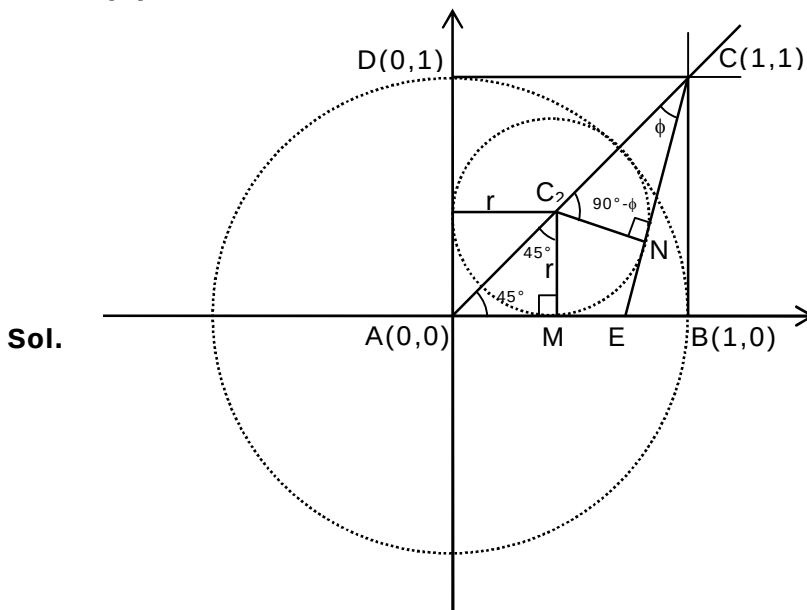
$$\&a - c = 0$$

$$\& \frac{a}{2} + \frac{b}{2} + \frac{c}{2} = 2$$

$$\Rightarrow a + b + c = 4$$

7. Let ABCD be a square of side of unit length. Let a circle C_1 centered at A with unit radius is drawn. Another circle C_2 which touches C_1 and the lines AD and AB are tangent to it, is also drawn. Let a tangent line from the point C to the circle C_2 meet the side AB at E. If the length of EB is $\alpha + \sqrt{3}\beta$, where α, β are integers, then $\alpha + \beta$ is equal to ____

Ans. (1)



$$(i) \sqrt{2}r + r = 1$$

$$r = \frac{1}{\sqrt{2} + 1}$$

$$r = \sqrt{2} - 1$$

$$(ii) CC_2 = 2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$$

$$\text{From } \triangle CC_2N = \sin \phi = \frac{\sqrt{2} - 1}{2(\sqrt{2} - 1)}$$

$$\phi = 30^\circ$$

(iii) In $\triangle ACE$ are sine law

$$\frac{AE}{\sin \phi} = \frac{AC}{\sin 105^\circ}$$

$$AE = \frac{1}{2} \times \frac{\sqrt{2}}{\sqrt{3} + 1} \cdot 2\sqrt{2}$$

$$AE = \frac{2}{\sqrt{3} + 1} = \sqrt{3} - 1$$

$$\therefore EB = 1 - (\sqrt{3} - 1)$$

$$2 - \sqrt{3}$$

$$\alpha = 2, \beta = -1 \Rightarrow \alpha + \beta = 1$$

8. Let z and ω be two complex numbers such that $\omega = z\bar{z} - 2z + 2$, $\left| \frac{z+i}{z-3i} \right| = 1$ and $\operatorname{Re}(\omega)$ has minimum value. Then, the minimum value of $n \in \mathbb{N}$ for which ω^n is real, is equal to _____

Ans. (4)

Sol. Let $z = x + iy$

$$|z + i| = |z - 3i|$$

$$\Rightarrow y = 1$$

$$\text{Now } \omega = x^2 + y^2 - 2x - 2iy + 2$$

$$\omega = x^2 + 1 - 2x - 2i + 2$$

$$\operatorname{Re}(\omega) = x^2 - 2x + 3$$

$$\operatorname{Re}(\omega) = (x-1)^2 + 2$$

$$\operatorname{Re}(\omega)_{\min} \text{ at } x = 1 \Rightarrow z = 1 + i$$

$$\text{Now } \omega = 1 + 1 - 2 - 2i + 2$$

$$\omega = 2(1-i) = 2\sqrt{2}e^{i\left(\frac{-\pi}{4}\right)}$$

$$\omega^n = 2\sqrt{2}e^{i\left(\frac{-n\pi}{4}\right)}$$

$$\text{If } \omega^n \text{ is real } \Rightarrow n = 4$$

9. Let $P = \begin{bmatrix} -30 & 20 & 56 \\ 90 & 140 & 112 \\ 120 & 60 & 14 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 7 & \omega^2 \\ -1 & -\omega & 1 \\ 0 & -\omega & -\omega + 1 \end{bmatrix}$ where $\omega = \frac{-1 + i\sqrt{3}}{2}$, and I_3 be the identity

matrix of order 3. If the determinant of the matrix $(P^{-1}AP - I_3)^2$ is $\alpha\omega^2$, then the value of α is equal to _____

Ans. (36)

Sol. $|P^{-1}AP - I|^2$

$$= |(P^{-1}AP - I)(P^{-1}AP - I)|^2$$

$$= |P^{-1}APP^{-1}AP - 2P^{-1}AP + I|$$

$$= |P^{-1}A^2P - 2P^{-1}AP + P^{-1}IP|$$

$$= |P^{-1}(A^2 - 2A + I)P|$$

$$= |P^{-1}(A - I)^2P|$$

$$= |P^{-1}||A - I|^2|P|$$

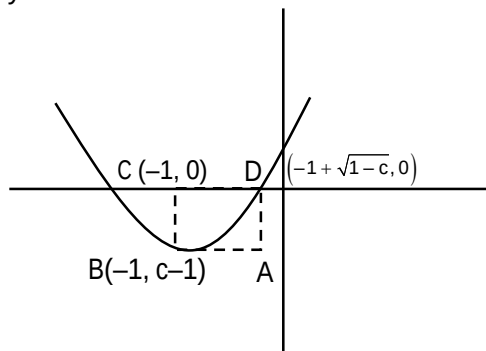
$$= |A - I|^2$$

$$\begin{aligned}
&= \begin{vmatrix} 1 & 7 & \omega^2 \\ -1 & -\omega-1 & 1 \\ 0 & -\omega & -\omega \end{vmatrix} \\
&= (1(\omega(\omega+1)+\omega)-7\omega+\omega^2.\omega)^2 \\
&= (\omega^2+2\omega-7\omega+1)^2 \\
&= (\omega^2-5\omega+1)^2 \\
&= (-6\omega)^2 \\
&= 36\omega^2 \Rightarrow \alpha = 36
\end{aligned}$$

- 10.** Let the curve $y=y(x)$ be the solution of the differential equation, $\frac{dy}{dx} = 2(x+1)$. If the numerical value of area bounded by the curve $y=y(x)$ and x-axis is $\frac{4\sqrt{8}}{3}$, then the value of $y(1)$ is equal to

Ans. (2)

Sol. $y = x^2 + 2x + c$



$$\text{Area of rectangle (ABCD)} = |(c-1)(\sqrt{1-c})|$$

$$\text{Area of parabola and x-axis} = 2 \left(\frac{2}{3} ((1-c)^{3/2}) \right) = \frac{4\sqrt{8}}{3}$$

$$1-c = 2 \Rightarrow c = -1$$

$$\text{Equation of } f(x) = x^2 + 2x - 1$$

$$f(1) = 1 + 2 - 1 = 2$$