

SECTION – A

1. The least value of $|z|$ where z is complex number which satisfies the inequality $\exp$

$$\left(\frac{(|z|+3)(|z|-1)}{|z|+1} \log_e 2 \right) \geq \log_{\sqrt{2}} |5\sqrt{7} + 9i|, i = \sqrt{-1} \text{ is equal to :}$$

- (1) 2
- (2) 3
- (3) 8
- (4) $\sqrt{5}$

Ans. (2)

Sol. $2^{\frac{(|z|+3)(|z|-1)}{(|z|+1)}} \geq 2^3 \Rightarrow \frac{(|z|+3)(|z|-1)}{|z|+1} \geq 3$

$$\Rightarrow |z|^2 + 2|z| - 3 \geq 3|z| + 3$$

$$\Rightarrow |z|^2 - |z| - 6 \geq 0$$

$$(|z|-3)(|z|+2) \geq 0$$

$$|z|_{\min} = 3$$

2. Let $f : S \rightarrow S$ where $S = (0, \infty)$ be a twice differentiable function such that $f(x+1) = xf(x)$. If $g : S \rightarrow \mathbb{R}$ be defined as $g(x) = \log_e f(x)$, then the value of $|g''(5) - g''(1)|$ is equal to :

- (1) $\frac{197}{144}$
- (2) $\frac{187}{144}$
- (3) $\frac{205}{144}$
- (4) 1

Ans. (3)

Sol. $f(x+1) = xf(x)$

$$g(x+1) = \log_e(f(x+1))$$

$$g(x+1) = \log_e x + \log f(x)$$

$$g(x+1) - g(x) = \log_e x$$

$$g''(x+1) - g''(x) = -\frac{1}{x^2}$$

$$g''(2) - g''(1) = -1$$

$$g''(3) - g''(2) = -\frac{1}{4}$$

$$g''(4) - g''(3) = -\frac{1}{9}$$

$$g''(5) - g''(4) = -\frac{1}{16}$$

$$g''(5) - g''(1) = -\left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}\right]$$

$$|g''(5) - g''(1)| = \left[\frac{144 + 36 + 16 + 9}{16 \times 9}\right] = \left[\frac{205}{16 \times 9}\right]$$

3. If $y = y(x)$ is the solution of the differential equation $\frac{dy}{dx} + (\tan x)y = \sin x$, $0 \leq x \leq \frac{\pi}{3}$, with $y(0) = 0$, then $y\left(\frac{\pi}{4}\right)$ equal to :

- (1) $\log_e 2$
- (2) $\frac{1}{2} \log_e 2$
- (3) $\left(\frac{1}{2\sqrt{2}}\right) \log_e 2$
- (4) $\frac{1}{4} \log_e 2$

Ans. (3)

Sol. I.f. = $e^{\int \tan x dx}$

$$= e^{\ln[\sec x]}$$

$$= \sec x$$

Solution of the equation

$$y(\sec x) = \int (\sin x)(\sec x) dx$$

$$\Rightarrow \frac{y}{\cos x} = \ln(\sec x) + c$$

Put $x = 0$, $c = 0$

$$\therefore y = \cos x \ln(\sec x)$$

put $x = \pi/4$

$$y = \frac{1}{\sqrt{2}} \ln \sqrt{2} = \frac{1}{2\sqrt{2}} \ln 2$$

$$y = \frac{\ln 2}{2\sqrt{2}}$$

- 4.** If the foot of the perpendicular from point (4, 3, 8) on the line $L_1 : \frac{x-a}{\ell} = \frac{y-2}{3} = \frac{z-b}{4}$, $\ell \neq 0$ is (3, 5, 7), then the shortest distance between the line L_1 and line $L_2 : \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$ is equal to :

(1) $\sqrt{\frac{2}{3}}$

(2) $\frac{1}{\sqrt{3}}$

(3) $\frac{1}{2}$

(4) $\frac{1}{\sqrt{6}}$

Ans. (4)

Sol. (3, 5, 7) lie on given line L_1

$$\frac{3-a}{\ell} = \frac{3}{3} = \frac{7-b}{4}$$

$$\frac{7-b}{4} = 1 \Rightarrow b = 3$$

$$\frac{3-a}{\ell} = 1 \Rightarrow 3-a = \ell$$

A (4, 3, 8)

B (3, 5, 7)

DR'S of AB = (1, -2, 1)

AB $\perp$ line L_1

$$(1)(\ell) + (-2)(3) + 4(1) = 0$$

$$\Rightarrow \ell = 2$$

$$a = 1$$

$$a = 1, b = 3, \ell = 2$$

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

$$\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$$

$$\text{S.D.} = \frac{\begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}}{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}} = \frac{1}{\sqrt{6}}$$

5. If (x, y, z) be an arbitrary point lying on a plane P which passes through the points $(42, 0, 0)$, $(0, 42, 0)$ and $(0, 0, 42)$, then the value of the expression

$$3 + \frac{x-11}{(y-19)^2(z-12)^2} + \frac{y-19}{(x-11)^2(z-12)^2} + \frac{z-12}{(z-11)^2(y-19)^2} - \frac{x+y+z}{14(x-11)(y-19)(z-12)} \text{ is equal}$$

to :

- (1) 3
- (2) 0
- (3) 39
- (4) -45

Ans. (1)

Sol. equation of plane $x + y + z = 42$

Let pt. on plane $x = 10, y = 21, z = 11$

$$3 + \frac{(-1)}{(4)(1)} + \frac{(2)}{(1)(1)} + \frac{(-1)}{(1)(4)} - \frac{42}{14(-1)(2)(-1)}$$

$$3 - \frac{1}{4} + 2 - \frac{1}{4} - \frac{3}{2} = 3$$

6. Consider the integral

$$I = \int_0^{10} \frac{[x]e^{[x]}}{e^{x-1}} dx$$

Where $[x]$ denotes the greatest integer less than or equal to x . Then the value of I is equal to :

- (1) $45(e-1)$
- (2) $45(e+1)$
- (3) $9(e-1)$
- (4) $9(e+1)$

Ans. (1)

Sol. $I = \int_0^{10} [x] \cdot e^{[x]+1-x} dx$

$$= \int_1^2 e^{2-x} dx + \int_2^3 2 \cdot e^{3-x} dx + \int_3^4 3 \cdot e^{4-x} dx + \dots + \int_9^{10} 9e^{10-x} dx$$
$$= -\{(1 - e) + 2(1 - e) + 3(1 - e) + \dots + 9(1 - e)\}$$
$$= 45(e - 1)$$

7. Let A (-1, 1), B (3, 4) and C(2, 0) be given three points. A line $y = mx$, $m > 0$, intersects lines AC and BC at point P and Q respectively. Let A_1 and A_2 be the areas of $\triangle ABC$ and $\triangle PQC$ respectively, such that $A_1 = 3A_2$, then the value of m is equal to :

(1) $\frac{4}{15}$

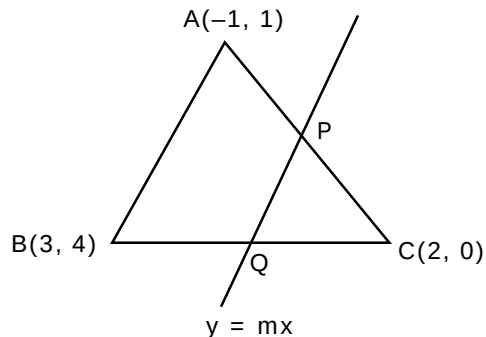
(2) 1

(3) 2

(4) 3

Ans. (2)

Sol.



$$A_1 = \triangle ABC = \frac{1}{2} \begin{vmatrix} -1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 4 & 1 \end{vmatrix}$$

$$A_1 = \frac{13}{2}$$

Equation of line AC is $y - 1 = -\frac{1}{3}(x + 1)$

solve it with line $y = mx$, we get $P\left(\frac{2}{3m+1}, \frac{2m}{3m+1}\right)$

Equation of line BC is $y - 0 = 4(x - 2)$

Solve it with line $y = mx$, we get $Q\left(\frac{-8}{m-4}, \frac{-8m}{m-4}\right)$

$$A_2 = \text{Area of } \triangle PQC = \frac{1}{2} \begin{vmatrix} 2 & 0 & 1 \\ \frac{2}{3m+1} & \frac{2m}{3m+1} & 1 \\ \frac{-8}{m-4} & \frac{-8m}{m-4} & 1 \end{vmatrix} = \frac{A_1}{3} = \frac{13}{6}$$

$$= \frac{1}{2} \left(2 \left(\frac{2m}{3m+1} + \frac{8m}{m-4} \right) - 1 \left(\frac{-16m}{(3m+1)(m-4)} + \frac{16m}{(3m+1)(m-4)} \right) \right)$$

$$= \pm \frac{13}{6}$$

$$\frac{26m^2}{3m^2 - 11m - 4} = \pm \frac{13}{6}$$

$$\Rightarrow 12m^2 = \pm (3m^2 - 11m - 4)$$

taking +ve sign

$$9m^2 + 11m + 4 = 0 \text{ (Rejected } \because m \text{ is imaginary)}$$

taking -ve sing

$$15m^2 - 11m - 4 = 0$$

$$m = 1, -\frac{4}{15}$$

8. Let f be a real valued function, defined on $\mathbb{R} - \{-1, 1\}$ and given by

$$f(x) = 3 \log_e \left| \frac{x-1}{x+1} \right| - \frac{2}{x-1}.$$

Then in which of the following intervals, function $f(x)$ is increasing ?

$$(1) (-\infty, -1) \cup \left(\left[\frac{1}{2}, \infty \right) - \{1\} \right)$$

$$(2) \left(-1, \frac{1}{2} \right]$$

$$(3) (-\infty, \infty) - \{-1, 1\}$$

$$(4) \left(-\infty, \frac{1}{2} \right] - \{-1\}$$

Ans. (1)

Sol.
$$f'(x) = \left(\frac{x+1}{x-1} \times \frac{x+1-(x-1)}{(x+1)^2} \right) 3 + \frac{2}{(x-1)^2} = \frac{6}{(x-1)(x+1)} + \frac{2}{(x-1)^2}$$

$$= \frac{2}{(x-1)} \left(\frac{3}{x+1} + \frac{1}{x+1} \right) = \frac{4(2x-1)}{(x+1)(x-1)^2}$$

$$\frac{\boxed{+} \quad \boxed{-} \quad \boxed{+}}{\frac{-1}{-1} \quad \frac{1}{2}}$$

$$x \in (-\infty, -1) \cup \left[\frac{1}{2}, \infty \right) - \{1\}$$

- 9.** Let the lengths of intercepts on x-axis and y-axis made by the circle $x^2 + y^2 + ax + 2ay + c = 0$, ($a < 0$) be $2\sqrt{2}$ and $2\sqrt{5}$, respectively. Then the shortest distance from origin to a tangent to this circle which is perpendicular to the line $x + 2y = 0$, is equal to :

- (1) $\sqrt{10}$
- (2) $\sqrt{6}$
- (3) $\sqrt{11}$
- (4) $\sqrt{7}$

Ans. (2)

Sol. $2\sqrt{a^2/4 - c} = 2\sqrt{2}$

$$\sqrt{a^2 - 4c} = 2\sqrt{2}$$

$$a^2 - 4c = 8 \quad \dots (1)$$

$$2\sqrt{a^2 - c} = 2\sqrt{5}$$

$$a^2 - c = 5 \quad \dots (2)$$

$$(2) - (1)$$

$$3c = -3a \Rightarrow c = -a$$

$$a^2 = 4 \Rightarrow a = -2$$

$$x^2 + y^2 - 2x - 4y - 1 = 0$$

$$\text{Equation of tangent } 2x - y + \lambda = 0$$

$$\therefore p = r$$

$$\left| \frac{2 - 2 + \lambda}{\sqrt{5}} \right| = \sqrt{6}$$

$$\Rightarrow \lambda = \pm\sqrt{30}$$

$$\therefore \text{tangent } 2x - y \pm \sqrt{30} = 0$$

$$\text{Distance from origin} = \frac{\sqrt{30}}{\sqrt{5}} = \sqrt{6}$$

10. Let A denote the event that a 6-digit integer formed by 0, 1, 2, 3, 4, 5, 6 without repetitions, be divisible by 3. Then probability of event A is equal to :

(1) $\frac{4}{9}$

(2) $\frac{9}{56}$

(3) $\frac{3}{7}$

(4) $\frac{11}{27}$

Ans. (1)

Sol. Total case = $6\bar{6}$

Fav. case = (0, 1, 2, 3, 4, 5) + (0, 1, 2, 4, 5, 6) + (1, 2, 3, 4, 5, 6)

$$= 5\bar{5} + 5\bar{5} + \bar{6}$$

$$= 1920$$

$$\text{Probability} = \frac{1920}{6\bar{6}} = \frac{4}{9}$$

11. Let $\alpha \in \mathbb{R}$ be such that the function $f(x) = \begin{cases} \frac{\cos^{-1}(1 - \{x\}^2) \sin^{-1}(1 - \{x\})}{\{x\} - \{x\}^3} & x \neq 0 \\ \alpha & x = 0 \end{cases}$ is

Continuous at $x = 0$, where $\{x\} = x - [x]$, $[x]$ is the greatest integer less than or equal to x .

Then :

(1) $\alpha = \frac{\pi}{4}$

(2) No such α exists

(3) $\alpha = 0$

(4) $\alpha = \frac{\pi}{\sqrt{2}}$

Ans. (2)

Sol. RHL = $\lim_{x \rightarrow 0^+} \frac{\cos^{-1}(1 - x^2) \sin^{-1}(1 - x)}{x(1 - x^2)} = \frac{\pi}{2} \lim_{x \rightarrow 0^+} \frac{\cos^{-1}(1 - x^2)}{x}$

= $\frac{\pi}{2} \lim_{x \rightarrow 0^+} \frac{-1}{\sqrt{1 - (1 - x^2)^2}} (-2x)$ (L' Hospital Rule)

= $\pi \lim_{x \rightarrow 0^+} \frac{x}{\sqrt{2x^2 - x^4}} = \pi \lim_{x \rightarrow 0^+} \frac{1}{\sqrt{2 - x^2}} = \frac{\pi}{\sqrt{2}}$

LHL = $\lim_{x \rightarrow 0^-} \frac{\cos^{-1}(1 - (1 + x)^2) \sin^{-1}(-x)}{(1 + x) - (1 + x)^3} = \frac{\pi}{2} \lim_{x \rightarrow 0^-} \frac{\sin^{-1} x}{(1 + x)[(1 + x)^2 - 1]} = \frac{\pi}{2} \lim_{x \rightarrow 0^-} \frac{\sin^{-1} x}{x^2 + 2x}$

= $\frac{\pi}{2} \left(\frac{1}{2} \right) = \frac{\pi}{4}$

As LHL $\neq$ RHL so $f(x)$ is not continuous at $x = 0$

12. The maximum value of $f(x) = \begin{vmatrix} \sin^2 x & 1 + \cos^2 x & \cos 2x \\ 1 + \sin^2 x & \cos^2 x & \cos 2x \\ \sin^2 x & \cos^2 x & \sin 2x \end{vmatrix}$, $x \in \mathbb{R}$ is :

(1) $\sqrt{7}$

(2) $\sqrt{5}$

(3) 5

(4) $\frac{3}{4}$

Ans. (2)

Sol. $C_1 \rightarrow C_1 + C_2$

$$\begin{vmatrix} 2 & 1 + \cos^2 x & \cos 2x \\ 2 & \cos^2 x & \cos 2x \\ 1 & \cos^2 x & \sin 2x \end{vmatrix}$$

$R_1 \rightarrow R_1 - R_2$

$$\begin{vmatrix} 0 & 1 & 0 \\ 2 & \cos^2 x & \cos 2x \\ 1 & \cos^2 x & \sin 2x \end{vmatrix}$$

$$= (-1)[2\sin 2x - \cos 2x] = \cos 2x - 2\sin 2x$$

$$\text{maximum value} = \sqrt{5}$$

- 13.** Consider a rectangle ABCD having 5, 7, 6, 9 points in the interior of the line segments AB, CD, BC, DA respectively. Let α be the number of triangles having these points from different sides as vertices and β be the number of quadrilaterals having these points from different sides as vertices. Then $(\beta - \alpha)$ is equal to:

- (1) 1890
- (2) 795
- (3) 717
- (4) 1173

Ans. (3)

Sol. $\alpha = {}^6C_1 {}^7C_1 {}^9C_1 + {}^5C_1 {}^7C_1 {}^9C_1 + {}^5C_1 {}^6C_1 {}^9C_1 + {}^5C_1 {}^6C_1 {}^7C_1 = 378 + 315 + 270 + 210 = 1173$

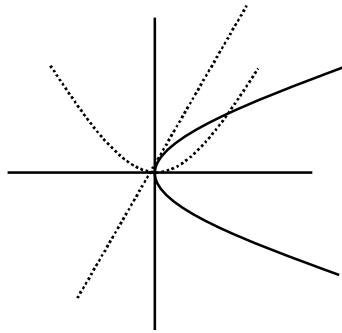
$$\beta = {}^5C_1 {}^6C_1 {}^7C_1 {}^9C_1 = 1890$$

$$\Rightarrow \beta - \alpha = 1890 - 1173 = 717$$

- 14.** Let C be the locus of the mirror image of a point on the parabola $y^2 = 4x$ with respect to the line $y = x$. Then the equation of tangent to C at P(2, 1) is :

- (1) $2x + y = 5$
- (2) $x + 2y = 4$
- (3) $x + 3y = 5$
- (4) $x - y = 1$

Ans. (4)



Sol. Image of $y^2 = 4x$ w.r.t. $y = x$ is $x^2 = 4y$
tangent from $(2, 1)$

$$xx_1 = 2(y + y_1)$$

$$2x = 2(y + 1)$$

$$x = y + 1$$

15. Given that the inverse trigonometric functions take principal values only. Then, the number of real values of x which satisfy $\sin^{-1}\left(\frac{3x}{5}\right) + \sin^{-1}\left(\frac{4x}{5}\right) = \sin^{-1} x$ is equal to :

(1) 1

(2) 2

(3) 3

(4) 0

Ans. (3)

Sol. Taking sine both sides

$$\frac{3x}{5} \sqrt{1 - \frac{16x^2}{25}} + \frac{4x}{5} \sqrt{1 - \frac{9x^2}{25}} = x$$

$$\Rightarrow 3x\sqrt{25 - 16x^2} = 25x - 4x\sqrt{25 - 9x^2}$$

$$\Rightarrow x = 0 \text{ or } 3\sqrt{25 - 16x^2} = 25 - 4\sqrt{25 - 9x^2}$$

$$\Rightarrow 9(25 - 16x^2) = 625 - 200\sqrt{25 - 9x^2} + 16(25 - 9x^2)$$

$$\Rightarrow 200\sqrt{25 - 9x^2} = 800$$

$$\Rightarrow \sqrt{25 - 9x^2} = 4$$

$$\Rightarrow x^2 = 1$$

$$\Rightarrow x = \pm 1$$

$\therefore$ Total number of solution = 3

- 16.** Let C_1 be the curve obtained by solution of differential equation $2xy \frac{dy}{dx} = y^2 - x^2, x > 0$ Let the curve C_2 be the solution of $\frac{2xy}{x^2 - y^2} = \frac{dy}{dx}$, If both the curves pass through $(1, 1)$ then the area enclosed by the curves C_1 and C_2 is equal to :

(1) $\frac{\pi}{2} - 1$

(2) $\frac{\pi}{4} + 1$

(3) $\pi - 1$

(4) $\pi + 1$

Ans. (1)

Sol.

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

Put $y = vx$

$$v + x \frac{dv}{dx} = \frac{v^2 x^2 - x^2}{2vx^2} = \frac{v^2 - 1}{2v}$$

$$x \frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v} = -\frac{(v^2 + 1)}{2v}$$

$$\Rightarrow \frac{2v}{v^2 + 1} dv = -\frac{dx}{x}$$

$$\ln(v^2 + 1) = -\ln x + \ln c \Rightarrow v^2 + 1 = \frac{c}{x}$$

$$\Rightarrow \frac{y^2}{x^2} + 1 = \frac{c}{x} \Rightarrow x^2 + y^2 = cx$$

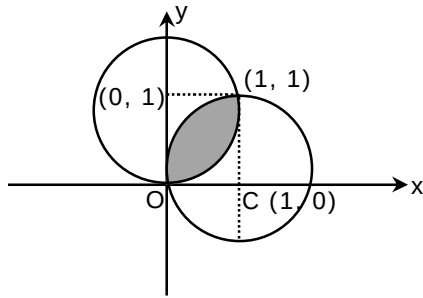
If pass through $(1,1)$

$$\therefore x^2 + y^2 - 2x = 0$$

Similarly for second differential equation $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

Equation of curve is $x^2 + y^2 - 2y = 0$

Now required area is



$$= \left(\frac{1}{4} \times \pi \times 1^2 - \frac{1}{2} \times 1 \times 1 \right) \times 2$$

$$= \left(\frac{\pi}{2} - 1 \right) \text{ sq. units}$$

17. Let $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 2\hat{i} - 3\hat{j} + 5\hat{k}$. If $\vec{r} \times \vec{a} = \vec{b} \times \vec{r}$, $\vec{r} \cdot (\alpha\hat{i} + 2\hat{j} + \hat{k}) = 3$ and

$\vec{r} \cdot (2\hat{i} + 5\hat{j} - \alpha\hat{k}) = -1, \alpha \in \mathbb{R}$, then the value of $\alpha + |\vec{r}|^2$ is equal to :

- (1) 11
- (2) 15
- (3) 9
- (4) 13

Ans. (2)

Sol. $\vec{r} \times \vec{a} = -\vec{r} \times \vec{b}$

$$\vec{r} \times (\vec{a} + \vec{b}) = 0 \quad (\vec{a} + \vec{b} = 3\hat{i} - \hat{j} + 2\hat{k})$$

$$\vec{r} \parallel (\vec{a} + \vec{b})$$

$$\vec{r} = \lambda (\vec{a} + \vec{b})$$

$$\therefore \vec{r} \cdot (2\hat{i} + 5\hat{j} - \alpha\hat{k}) = -1$$

$$\lambda [3\hat{i} - \hat{j} + 2\hat{k}] \cdot [2\hat{i} + 5\hat{j} - \alpha\hat{k}] = -1$$

$$\Rightarrow \lambda(6 - 5 - 2\alpha) = -1$$

$$\lambda(1 - 2\alpha) = -1 \quad \dots(1)$$

$$\vec{r} \cdot (\alpha\hat{i} + 2\hat{j} + \hat{k}) = 3$$

$$\lambda(3\hat{i} - \hat{j} + 2\hat{k}) \cdot (\alpha\hat{i} + 2\hat{j} + \hat{k}) = 3$$

$$\Rightarrow \lambda[3\alpha - 2 + 2] = 3 \Rightarrow \lambda\alpha = 1 \quad \dots(2)$$

(1) & (2)

$$\lambda \left[1 - \frac{2}{\lambda} \right] = -1$$

$$\lambda - 2 = -1 \Rightarrow \lambda = 1 \quad \alpha = 1$$

$$\vec{r} = 3\hat{i} - \hat{j} + 2\hat{k}$$

$$\alpha + |\vec{r}|^2 \Rightarrow 1 + 14 = 15$$

- 18.** Let $P(x) = x^2 + bx + c$ be a quadratic polynomial with real coefficients such that $\int_0^1 P(x) dx = 1$ and $P(x)$ leaves remainder 5 when it is divided by $(x - 2)$. Then the value of $9(b+c)$ is equal to :

- (1) 7
(2) 11
(3) 15
(4) 9

Ans. (1)

Sol. $(x - 2)Q(x) + 5 = x^2 + bx + c$

Put $x = 2$

$$5 = 2b + c + 4 \quad \dots(1)$$

$$\int_0^1 (x^2 + bx + c) dx = 1$$

$$\Rightarrow \frac{1}{3} + \frac{b}{2} + c = 1$$

$$\frac{b}{2} + c = \frac{2}{3} \quad \dots(2)$$

Solve (1) & (2)

$$b = \frac{2}{9}$$

$$c = \frac{5}{9}$$

$$9(b+c) = 7$$

19. If the points of intersections of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the circle $x^2 + y^2 = 4b$, $b > 4$ lie on the curve $y^2 = 3x^2$, then b is equal to :

- (1) 5
- (2) 6
- (3) 12
- (4) 10

Ans. (3)

Sol. $\frac{x^2}{16} + \frac{y^2}{b^2} = 1 \quad \dots(1)$

$x^2 + y^2 = 4b \quad \dots(2)$

$y^2 = 3x^2 \quad \dots(3)$

From eq (2) and (3) $x^2 = b$ and $y^2 = 3b$

From equation (1) $\frac{b}{16} + \frac{3b}{b^2} = 1$

$\Rightarrow b^2 + 48 = 16b$

$\Rightarrow b = 12$

20. Let $A = \{2, 3, 4, 5, \dots, 30\}$ and ' $\simeq$ ' be an equivalence relation on $A \times A$, defined by $(a, b) \simeq (c, d)$, if and only if $ad = bc$. Then the number of ordered pairs which satisfy this equivalence relation with ordered pair $(4, 3)$ is equal to :

- (1) 7
- (2) 5
- (3) 6
- (4) 8

Ans. (1)

Sol. $ad = bc$

$(a, b) R (4, 3) \Rightarrow 3a = 4b$

$a = \frac{4}{3}b$

b must be multiple of 3

$b = \{3, 6, 9, \dots, 30\}$

$(a, b) = (4, 3), (8, 6), (12, 9), (16, 12), (20, 15), (24, 18), (28, 21)$

$\Rightarrow 7$ ordered pair

SECTION – B

1. Let $\vec{c}$ be a vector perpendicular to the vectors $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$. If $\vec{c} \cdot (\hat{i} + \hat{j} + 3\hat{k}) = 8$ then the value of $\vec{c} \cdot (\vec{a} \times \vec{b})$ is equal to

Ans. (28)

Sol. $\vec{c} \cdot (\vec{a} \times \vec{b}) = [\vec{c} \vec{a} \vec{b}]$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = (3, -2, 1)$$

$$\vec{c} \perp \vec{a}, \vec{c} \perp \vec{b} \Rightarrow \vec{c} \parallel \vec{a} \times \vec{b}$$

$$\vec{c} = \lambda (\vec{a} \times \vec{b})$$

$$\Rightarrow \vec{c} = \lambda (3\hat{i} - 2\hat{j} + \hat{k})$$

$$\vec{c} \cdot (\hat{i} + \hat{j} + 3\hat{k}) = 8$$

$$\Rightarrow 3\lambda - 2\lambda + 3\lambda = 8$$

$$\Rightarrow 4\lambda = 8 \Rightarrow \lambda = 2$$

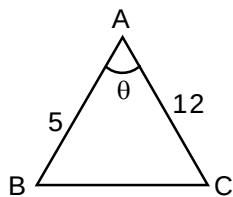
$$\vec{c} = 6\hat{i} - 4\hat{j} + 2\hat{k}$$

$$\vec{c} \cdot (\vec{a} \times \vec{b}) = [\vec{c} \vec{a} \vec{b}] = \begin{vmatrix} 6 & -4 & 2 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$\Rightarrow 18 + 8 + 2 = 28$$

2. In $\triangle ABC$, the lengths of sides AC and AB are 12 cm and 5 cm, respectively. If the area of $\triangle ABC$ is 30 cm^2 and R and r are respectively the radii of circumcircle and incircle of $\triangle ABC$, then the value of $2R + r$ (in cm) is equal to

Ans. (15)

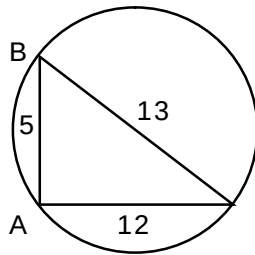


Sol.

$$\text{Area} = \frac{1}{2}(5)(12)\sin\theta = 30$$

$$\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

Δ is right angle Δ



$$r = (s-a) \tan \frac{A}{2}$$

$$r = (s-a)$$

$$r = (s-a) \quad (a = 2R)$$

$$2R + r = s$$

$$2R + r = \frac{30}{2} = 15$$

3. Consider the statistics of observations as follows:

	Size	Mean	Variance
Observation I	10	2	2
Observation II	n	3	1

If the variance of the combined set of these two observations is $\frac{17}{9}$, then the value of n is equal to

Ans. (5)

Sol. For group-1 : $\frac{\sum x_i}{10} = 2 \Rightarrow \sum x_i = 20$

$$\frac{\sum x_i^2}{10} - (2)^2 = 2 \Rightarrow \sum x_i^2 = 60$$

For group-2 : $\frac{\sum y_i}{n} = 3 \Rightarrow \sum y_i = 3n$

$$\frac{\sum y_i^2}{n} - 3^2 = 1 \Rightarrow \sum y_i^2 = 10n$$

Now, combined variance

$$\sigma^2 = \frac{\sum (x_i^2 + y_i^2)}{10 + n} - \left(\frac{\sum (x_i + y_i)}{10 + n} \right)^2$$

$$\Rightarrow \frac{17}{9} = \frac{60 + 10n}{10 + n} - \frac{(20 + 3n)^2}{(10 + n)^2}$$

$$\Rightarrow 17(n^2 + 20n + 100) = 9(n^2 + 40n + 200)$$

$$\Rightarrow 8n^2 - 20n - 100 = 0$$

$$\Rightarrow 2n^2 - 5n - 25 = 0 \Rightarrow n = 5$$

4. Let

$$S_n(x) = \log_{\frac{1}{a^2}} x + \log_{\frac{1}{a^3}} x + \log_{\frac{1}{a^6}} x + \log_{\frac{1}{a^{11}}} x + \log_{\frac{1}{a^{18}}} x + \log_{\frac{1}{a^{27}}} x + \dots \text{ up to } n\text{-terms,}$$

Where $a > 1$. If $S_{24}(x) = 1093$ and $S_{12}(2x) = 265$, the value of a is equal to

Ans. (16)

Sol. $S_n(x) = \log_a x^2 + \log_a x^3 + \log_a x^6 + \log_a x^{11}$

$$S_n(x) = 2\log_a x + 3\log_a x + 6\log_a x + 11\log_a x + \dots$$

$$S_n(x) = \log_a x(2 + 3 + 6 + 11 + \dots)$$

$$S_r = 2 + 3 + 6 + 11$$

$$\text{General term } T_r = r^2 - 2r + 3$$

$$S_n(x) = \sum_{r=1}^n \log_a x(r^2 - 2r + 3)$$

$$S_{24}(x) = \sum_{r=1}^{24} \log_a x(r^2 - 2r + 3)$$

$$S_{24}(x) = \log_a \sum_{r=1}^{24} (r^2 - 2r + 3)$$

$$1093 = 4372 \log_a x$$

$$\log_a x = \frac{1}{4}$$

$$x = a^{1/4}$$

$$S_{12}(2x) = \log_a(2x) \sum_{r=1}^{12} (r^2 - 2r + 3)$$

$$265 = 530 \log_a(2x)$$

$$\log_a(2x) = \frac{1}{2}$$

$$2x = a^{1/2}$$

After solving (i) and (ii), we get

$$a^{1/4} = 2$$

$$a = 16$$

5. Let n be a positive integer. Let $A = \sum_{k=0}^n (-1)^k {}^nC_k \left[\left(\frac{1}{2}\right)^k + \left(\frac{3}{4}\right)^k + \left(\frac{7}{8}\right)^k + \left(\frac{15}{16}\right)^k + \left(\frac{31}{32}\right)^k \right]$ If $63A = 1 - \frac{1}{2^{30}}$, then n is equal to

Ans. (6)

Sol. $A = \left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n + \left(\frac{1}{8}\right)^n + \left(\frac{1}{16}\right)^n + \left(\frac{1}{32}\right)^n$

$$= \frac{1}{2^n} + \frac{1}{2^{2n}} + \frac{1}{2^{3n}} + \frac{1}{2^{4n}} + \frac{1}{2^{5n}}$$

$$= \frac{1}{2^n} \left[\frac{1 - \left(\frac{1}{2^n}\right)^5}{1 - \frac{1}{2^n}} \right]$$

$$A = \frac{2^{5n} - 1}{2^{5n}(2^n - 1)}$$

$$63A = \frac{63(2^{5n} - 1)}{2^{5n}(2^n - 1)}$$

$$\frac{63}{2^n - 1} \left(1 - \frac{1}{2^{5n}}\right) = 63A = \left(1 - \frac{1}{2^{30}}\right)$$

$$= \frac{63}{2^n - 1} \left(1 - \frac{1}{2^{5n}}\right) = \left(1 - \frac{1}{2^{30}}\right)$$

$$n = 6$$

6. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} x + a, & x < 0 \\ |x - 1|, & x \geq 0 \end{cases} \text{ and } g(x) = \begin{cases} x + 1, & x < 0 \\ (x - 1)^2 + b, & x \geq 0 \end{cases}$$

Where a, b are non-negative real numbers. If $(g \circ f)(x)$ is continuous for all $x \in \mathbb{R}$, then $a + b$ is equal to

Ans. (1)

Sol.

$$g[f(x)] = \begin{cases} f(x) + 1 & f(x) < 0 \\ (f(x) - 1)^2 + b & f(x) \geq 0 \end{cases}$$

$$g[f(x)] = \begin{cases} x + a + 1 & x + a < 0 \text{ \& } x < 0 \\ |x - 1| + 1 & |x - 1| < 0 \text{ \& } x \geq 0 \\ (x + a - 1)^2 + b & x + a \geq 0 \text{ \& } x < 0 \\ (|x - 1| - 1)^2 + b & |x - 1| \geq 0 \text{ \& } x \geq 0 \end{cases}$$

$$g[f(x)] = \begin{cases} x + a + 1 & x \in (-\infty, -a) \text{ \& } x \in (-\infty, 0) \\ |x - 1| + 1 & x \in \phi \\ (x + a - 1)^2 + b & x \in [-a, \infty) \text{ \& } x \in (-\infty, 0) \\ (|x - 1| - 1)^2 + b & x \in \mathbb{R} \text{ \& } x \in [0, \infty) \end{cases}$$

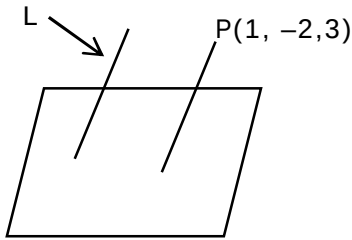
$$g[f(x)] = \begin{cases} x + a + 1 & x \in (-\infty, -a) \\ (x + a - 1)^2 + b & x \in [-a, 0) \\ (|x - 1| - 1)^2 + b & x \in [0, \infty) \end{cases}$$

$g(f(x))$ is continuous

$$\begin{aligned}
 &\text{at } x = -a \quad \& \quad \text{at } x = 0 \\
 &1 = b + 1 \quad \& \quad (a-1)^2 + b = b \\
 &b = 0 \quad \& \quad a = 1 \\
 &\Rightarrow a + b = 1
 \end{aligned}$$

7. If the distance of the point $(1, -2, 3)$ from the plane $x + 2y - 3z + 10 = 0$ measured parallel to the line, $\frac{x-1}{3} = \frac{2-y}{m} = \frac{z+3}{1}$ is $\sqrt{\frac{7}{2}}$, then the value of $|m|$ is equal to

Ans. (2)



Sol.

$$\frac{x-1}{3} = \frac{y+2}{-m} = \frac{z-3}{1} = \lambda$$

Pt. Q $(3\lambda + 1, -m\lambda - 2, \lambda + 3)$ lie on plane

$$(3\lambda + 1) + 2(-m\lambda - 2) - 3(\lambda + 3) + 10 = 0$$

$$\Rightarrow 3\lambda - 2m\lambda - 3\lambda + 1 - 4 - 9 + 10 = 0$$

$$\Rightarrow -2m\lambda = 2$$

$$m\lambda = -1 \Rightarrow \lambda = -\frac{1}{m}$$

$$Q\left[-\frac{3}{m} + 1, -1, -\frac{1}{m} + 3\right]$$

$$PQ = \sqrt{\frac{7}{2}}$$

$$\sqrt{\left(-\frac{3}{m}\right)^2 + 1 + \left(-\frac{1}{m}\right)^2} = \sqrt{\frac{7}{2}}$$

$$\Rightarrow \frac{10 + m^2}{m^2} = \frac{7}{2}$$

$$\Rightarrow 20 + 2m^2 = 7m^2$$

$$m^2 = 4 \Rightarrow |m| = 2$$

8. Let $\frac{1}{16}$, a and b be in G.P. and $\frac{1}{a}$, $\frac{1}{b}$, 6 be in A.P. where $a, b, > 0$. Then $72(a+b)$ is equal to

Ans. (14)

Sol. $a^2 = \frac{b}{16}$ and $\frac{2}{b} = \frac{1}{a} + 6$

Solving, we get $a = \frac{1}{12}$ or $a = -\frac{1}{4}$ [rejected]

$$\text{if } a = \frac{1}{12} \Rightarrow b = \frac{1}{9}$$

$$\therefore 72(a+b) = 72\left(\frac{1}{12} + \frac{1}{9}\right) = 14$$

9. Let $A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ be two 2×1 matrices with real entries such that $A = XB$, where $x = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 \\ 1 & k \end{bmatrix}$, and $k \in \mathbb{R}$. If $a_1^2 + a_2^2 = \frac{2}{3}(b_1^2 + b_2^2)$ and $(k^2 + 1)b_2^2 \neq -2b_1b_2$ then the value of k is

.....
Ans. (1)

Sol.

$$XB = A$$

$$\frac{1}{\sqrt{3}} \begin{bmatrix} b_1 - b_2 \\ b_1 + kb_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$b_1 - b_2 = \sqrt{3}a_1 \Rightarrow 3a_1^2 = b_1^2 + b_2^2 - 2b_1b_2$$

$$b_1 + kb_2 = \sqrt{3}a_2 \Rightarrow 3a_2^2 = b_1^2 + k^2b_2^2 + 2kb_1b_2$$

$$3(a_1^2 + a_2^2) = 2b_1^2 + (k^2 + 1)b_2^2 + 2b_1b_2(k - 1)$$

$$(k^2 - 1)b_2^2 + 2b_1b_2(k - 1)$$

$$(k - 1)((k + 1)b_2^2 + 2b_1b_2) = 0$$

$$\Rightarrow k = 1$$

10. For real number α, β, γ and δ , if

$$\int \frac{(x^2 - 1) + \tan^{-1}\left(\frac{x^2 + 1}{x}\right)}{(x^4 + 3x^2 + 1) \tan^{-1}\left(\frac{x^2 + 1}{x}\right)} dx$$

$$= \alpha \log_e \left(\tan^{-1}\left(\frac{x^2 + 1}{x}\right) \right) + \beta \tan^{-1}\left(\frac{\gamma(x^2 - 1)}{x}\right) + \delta \tan^{-1}\left(\frac{x^2 + 1}{x}\right) + C$$

Where C is an arbitrary constant, then the value of $10(\alpha + \beta\gamma + \delta)$ is equal to

Ans. (6)

Sol.
$$\int \frac{x^2 - 1}{(x^4 + 3x^2 + 1) \tan^{-1}\left(\frac{x^2 + 1}{x}\right)} dx + \int \frac{1}{x^4 + 3x^2 + 1} dx$$

$$\int \frac{1 - \frac{1}{x^2}}{\left[\left(x + \frac{1}{x} \right)^2 + 1 \right] \tan^{-1} \left(x + \frac{1}{x} \right)} dx + \int \frac{dx}{x^4 + 3x^2 + 1}$$

$\downarrow$
 I_1

$\downarrow$
 I_2

$$\tan^{-1} \left(x + \frac{1}{x} \right) = t$$

$$I_1 = \int \frac{dt}{t}$$

$$I_1 = \ln(t) = \ln \left| \tan^{-1} \left(x + \frac{1}{x} \right) \right|$$

Now

$$I_2 = \int \frac{dx}{x^4 + 3x^2 + 1}$$

$$= \frac{1}{2} \int \frac{(x^2 + 1) - (x^2 - 1)}{x^4 + 3x^2 + 1} dx$$

$$= \frac{1}{2} \left[\int \frac{1 + \frac{1}{x^2}}{x^2 + 3 + \frac{1}{x^2}} dx - \int \frac{\left(1 - \frac{1}{x^2} \right)}{x^2 + 3 + \frac{1}{x^2}} dx \right]$$

$$= \frac{1}{2} \left[\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x} \right)^2 + 5} dx - \int \frac{\left(1 - \frac{1}{x^2} \right)}{\left(x + \frac{1}{x} \right)^2 + 1} dx \right]$$

$\downarrow$
 $x - \frac{1}{x} = u$

$\downarrow$
 $x + \frac{1}{x} = v$

$$= \frac{1}{2} \left[\int \frac{du}{u^2 + (\sqrt{5})^2} - \int \frac{dv}{v^2 + 1} \right]$$

$$I_2 = \frac{1}{2\sqrt{5}} \tan^{-1} \left(\frac{x - \frac{1}{x}}{\sqrt{5}} \right) - \frac{1}{2} \tan^{-1} \left(x + \frac{1}{x} \right)$$

$$I = I_1 + I_2 = \ln \left| \tan^{-1} \left(x + \frac{1}{x} \right) \right| + \frac{1}{2\sqrt{5}} \ln \left(\frac{x^2 - 1}{\sqrt{5}x} \right) - \frac{1}{2} \tan^{-1} \left(\frac{x^2 + 1}{x} \right) + C$$

$$\alpha = 1, \beta = \frac{1}{2\sqrt{5}}, \lambda = \frac{1}{\sqrt{5}}, \delta = -\frac{1}{2}$$

$$10(\alpha + \beta\lambda + \delta) = 10\left[1 + \frac{1}{10} - \frac{1}{2}\right]$$

$$= 10\left(\frac{1}{10} + \frac{1}{2}\right)$$

$$= 1 + 5 = 6$$