

SECTION – A

1. Let the system of linear equations

$$4x + \lambda y + 2z = 0$$

$$2x - y + z = 0$$

$$\mu x + 2y + 3z = 0, \lambda, \mu \in \mathbb{R}$$

Has a non-trivial solution. Then which of the following is true?

(1) $\mu = 6, \lambda \in \mathbb{R}$

(2) $\mu = 2, \mu \in \mathbb{R}$

(3) $\mu = 3, \mu \in \mathbb{R}$

(4) $\mu = -6, \lambda \in \mathbb{R}$

Ans. (1)

Sol. For non trivial solution

$$\Delta = 0$$

$$\begin{vmatrix} 4 & \lambda & 2 \\ 2 & -1 & 1 \\ \mu & 2 & 3 \end{vmatrix} = 0$$

$$4(-3 - 2) - \lambda(6 - \mu) + 2(4 + \mu) = -20 - 6\lambda + \lambda\mu + 8 + 2\mu$$
$$= 12 - 6\lambda + \lambda\mu + 2\mu$$

$$\Rightarrow -12 - 6\lambda + (\lambda + 2)\mu$$

$$\mu = 6, \lambda \in \mathbb{R}$$

2. A pole stands vertically inside a triangular park ABC. Let the angle of elevation of the top of the pole from each corner of the park be $\frac{\pi}{3}$. If the radius of the circumcircle of $\triangle ABC$ is 2, then the height of the pole is equal to:

(1) $\frac{1}{\sqrt{3}}$

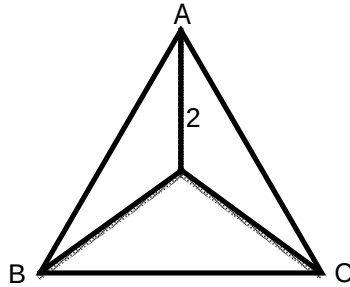
(2) $\sqrt{3}$

(3) $2\sqrt{3}$

(4) $\frac{2\sqrt{3}}{3}$

Ans. (3)

Sol.



$$\tan 60^\circ = \frac{h}{2} \Rightarrow h = 2\sqrt{3}$$

3. Let in a series of $2n$ observations, half of them are equal to a and remaining half are equal to $-a$. Also by adding a constant b in each of these observations, the mean and standard deviation of new set become 5 and 20, respectively. Then the value of $a^2 + b^2$ is equal to:

- (1) 250
- (2) 925
- (3) 650
- (4) 425

Ans. (4)

Sol. Given series

$(a, a, a, \dots, n \text{ times}), (-a, -a, -a, \dots, n \text{ times})$

$$\text{Now } \bar{x} = \frac{\sum x_i}{2n} = 0$$

as $x_i \rightarrow x_i + b$

then $\bar{x} \rightarrow \bar{x} + b$

So, $\bar{x} + b = 5 \Rightarrow b = 5$

No change in S.D. due to change in origin

$$\sigma = \sqrt{\frac{\sum x_i^2}{2n} - (\bar{x})^2} = \sqrt{\frac{2na^2}{2n} - 0}$$

$$20 = \sqrt{a^2} \Rightarrow a = 20$$

$$a^2 + b^2 = 425$$

4. Let $g(x) = \int_0^x f(t) dt$, where f is continuous function in $[0, 3]$ such that $\frac{1}{3} \leq f(t) \leq 1$ for all $t \in [0, 1]$ and $0 \leq f(t) \leq \frac{1}{2}$ for all $t \in (1, 3]$. The largest possible interval in which $g(3)$ lies is:
- (1) $[1, 3]$ (2) $\left[-1, -\frac{1}{2}\right]$ (3) $\left[-\frac{3}{2}, -1\right]$ (4) $\left[\frac{1}{3}, 2\right]$

Ans. (4)

Sol. $\int_0^1 \frac{1}{3} dt + \int_1^3 0 dt \leq g(3) \leq \int_0^1 1 dt + \int_1^3 \frac{1}{2} dt$

$$\frac{1}{3} \leq g(3) \leq 2$$

5. If $15 \sin^4 \alpha + 10 \cos^4 \alpha = 6$, for some $\alpha \in \mathbb{R}$, then the value of $27 \sec^6 \alpha + 8 \operatorname{cosec}^6 \alpha$ is equal to:
- (1) 250
(2) 500
(3) 400
(4) 350

Ans. (1)

Sol. $15 \sin^4 \theta + 10 \cos^4 \theta = 6$
 $\Rightarrow 15 \sin^4 \theta + 10(1 - \sin^2 \theta)^2 = 6$
 $\Rightarrow 25 \sin^4 \theta - 20 \sin^2 \theta + 4 = 0$
 $\Rightarrow (5 \sin^2 \theta - 2)^2 = 0 \Rightarrow \sin^2 \theta = \frac{2}{5}, \cos^2 \theta = \frac{3}{5}$
 Now $27 \operatorname{cosec}^6 \theta + 8 \sec^6 \theta = 27 \left(\frac{125}{27}\right) + 8 \left(\frac{125}{8}\right) = 250$

6. Let $f : \mathbb{R} - \{3\} \rightarrow \mathbb{R} - \{1\}$ be defined by $f(x) = \frac{x-2}{x-3}$.

Let $g : \mathbb{R} - \mathbb{R}$ be given as $g(x) = 2x - 3$. The, the sum of all the values of x for which $f^{-1}(x) + g^{-1}(x) = \frac{13}{2}$ is equal to

- (1) 7
(2) 5
(3) 2
(4) 3

Ans. (2)

Sol. $f^{-1}(x) + g^{-1}(x) = \frac{13}{2}$

$$\Rightarrow \frac{3x-2}{x-1} + \frac{x+3}{2} = \frac{13}{2}$$

$$\Rightarrow 2(3x-2) + (x-1)(x+3) = 13(x-1)$$

$$\Rightarrow x^2 - 5x + 6 = 0$$

$$\Rightarrow x = 2 \text{ or } 3$$

7. Let S_1 be the sum of first $2n$ terms of an arithmetic progression. Let S_2 be the sum of first $4n$ terms of the same arithmetic progression. If $(S_2 - S_1)$ is 1000, then the sum of the first $6n$ terms of the arithmetic progression is equal to :

(1) 3000

(2) 7000

(3) 5000

(4) 1000

Ans (1)

Sol. $S_{4n} - S_{2n} = 1000$

$$\Rightarrow \frac{4n}{2}(2a + (4n-1)d) - \frac{2n}{2}(2a + (2n-1)d) = 1000$$

$$\Rightarrow 2an + 6n^2d - nd = 1000$$

$$\Rightarrow \frac{6n}{2}(2a + (6n-1)d) = 3000$$

$$\therefore S_{6n} = 3000$$

8. Let $S_1 : x^2 + y^2 = 9$ and $S_2 : (x-2)^2 + y^2 = 1$. Then the locus of center of a variable circle S which touches S_1 internally and S_2 externally always passes through the points:

(1) $\left(\frac{1}{2}, \pm \frac{\sqrt{5}}{2}\right)$

(2) $\left(2, \pm \frac{3}{2}\right)$

(3) $(1, \pm 2)$

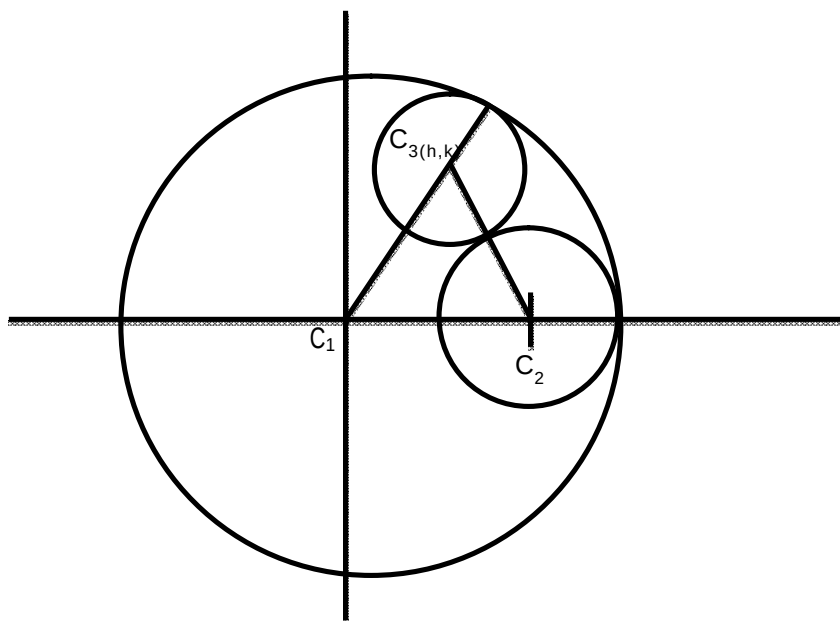
(4) $(0, \pm \sqrt{3})$

Ans. (2)

Sol. $C_1 : (0,0), r_1 = 3$

$$C_2 : (2, 0), r_2 = 1$$

Let centre of variable circle be $C_3(h,k)$ and radius be r .



$$C_3C_1 = 3 - r$$

$$C_2C_1 = 1 + r$$

$$C_3C_1 + C_2C_1 = 4$$

So locus is ellipse whose foci are C_1 & C_2

And major axis is $2a = 4$ and $2ae = C_1C_2 = 2$

$$\Rightarrow e = \frac{1}{2}$$

$$\Rightarrow b^2 = 4 \left(1 - \frac{1}{4} \right) = 3$$

Centre of ellipse is midpoint of C_1 & C_2 is $(1,0)$

$$\text{Equation of ellipse is } \frac{(x-1)^2}{2^2} + \frac{y^2}{(\sqrt{3})^2} = 1$$

Now by cross checking the option $\left(2, \pm \frac{3}{2} \right)$ satisfied it.

9. Let the centroid of an equilateral triangle ABC be at the origin. Let one of the sides of the equilateral triangle be along the straight line $x + y = 3$. If R and r be the radius of circumcircle and incircle respectively of $\triangle ABC$, then $(R + r)$ is equal to

(1) $2\sqrt{2}$

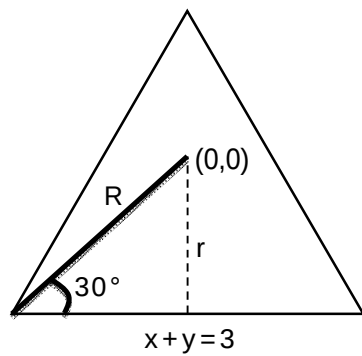
(2) $3\sqrt{2}$

(3) $7\sqrt{2}$

(4) $\frac{9}{\sqrt{2}}$

Ans. (4)

Sol.



$$r = \left| \frac{0+0-3}{\sqrt{2}} \right| = \frac{3}{\sqrt{2}}$$

$$\sin 30^\circ = \frac{r}{R} = \frac{1}{2}$$

$$R = 2r$$

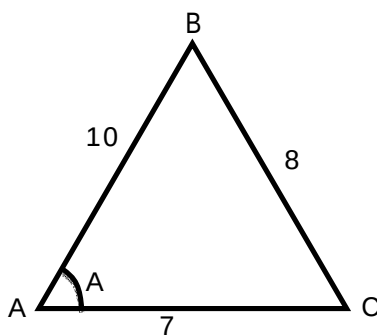
$$\text{So, } r + R = 3r = 3 \times \left(\frac{3}{\sqrt{2}} \right) = \frac{9}{\sqrt{2}}$$

10. In a triangle ABC, if $|\vec{BC}| = 8$, $|\vec{CA}| = 7$, $|\vec{AB}| = 10$, then the projection of the vector $\vec{AB}$ on $\vec{AC}$ is equal to:

- (1) $\frac{25}{4}$
- (2) $\frac{85}{14}$
- (3) $\frac{127}{20}$
- (4) $\frac{115}{16}$

Ans. (2)

Sol.



Projection of AB on AC is $= AB \cos A$

$$= 10 \cos A$$

By cosine rule

$$\begin{aligned}\cos A &= \frac{10^2 + 7^2 - 8^2}{2 \cdot 10 \cdot 7} \\ &= \frac{85}{140}\end{aligned}$$

$$\Rightarrow 10 \cos A = 10 \left(\frac{85}{140} \right) = \frac{85}{14}$$

- 11.** Let in a Binomial distribution, consisting of 5 independent trials, probabilities of exactly 1 and 2 successes be 0.4096 and 0.2048 respectively. Then the probability of getting exactly 3 successes is equal to:

- (1) $\frac{80}{243}$
- (2) $\frac{32}{625}$
- (3) $\frac{128}{625}$
- (4) $\frac{40}{243}$

Ans. (2)

Sol. ${}^5C_1 p^1 q^4 = 0.4096 \quad \dots(1)$

$${}^5C_2 p^2 q^3 = 0.2048 \quad \dots(2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{q}{2p} = 2 \Rightarrow q = 4p$$

$$p + q = 1 \Rightarrow p = \frac{1}{5}, q = \frac{4}{5}$$

$$\begin{aligned}P(\text{exactly } 3) &= {}^5C_3 (p)^3 (q)^2 = {}^5C_3 \left(\frac{1}{5} \right)^3 \left(\frac{4}{5} \right)^2 \\ &= 10 \times \frac{1}{125} \times \frac{16}{25} = \frac{32}{625}\end{aligned}$$

- 12.** Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors perpendicular to each other and $|\vec{a}| = |\vec{b}|$. If $|\vec{a} \times \vec{b}| = |\vec{a}|$, then the angle between the vectors $\left(\vec{a} + \vec{b} + \left(\vec{a} \times \vec{b}\right)\right)$ and $\vec{a}$ is equal to :

(1) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(2) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(3) $\sin^{-1}\left(\frac{1}{\sqrt{6}}\right)$

(4) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Ans. (2)

Sol. Given $|\vec{a} \times \vec{b}| = |\vec{a}| = |\vec{b}|$

$$\cos \theta = \frac{\vec{a} \cdot \left(\vec{a} + \vec{b} + \left(\vec{a} \times \vec{b}\right)\right)}{|\vec{a}| \cdot \left|\vec{a} + \vec{b} + \vec{a} \times \vec{b}\right|}$$

Let $|\vec{a}| = a$

$$\cos \theta = \frac{a^2 + 0 + 0}{a \times \sqrt{a^2 + a^2 + a^2}} = \frac{a^2}{a^2 \sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

- 13.** Let a complex number be $w = 1 - \sqrt{3}i$. Let another complex number z be such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$. Then the area of the triangle with vertices origin, z and w is equal to:

(1) $\frac{1}{2}$

(2) 4

(3) 2

(4) $\frac{1}{4}$

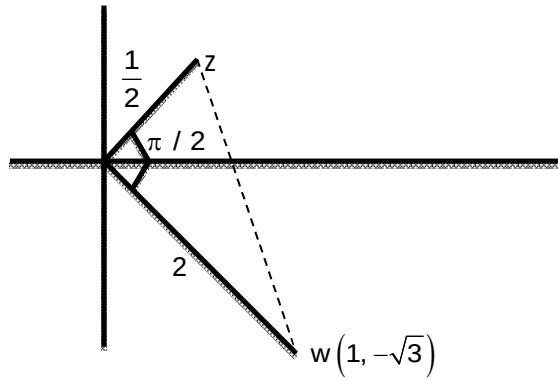
Ans. (1)

Sol. $w = 1 - \sqrt{3}i$

$$|w| = 2$$

$$|zw| = 1 \Rightarrow |z| = \frac{1}{|w|} = \frac{1}{2}$$

$$\arg(z) - \arg(w) = \pi / 2$$



$$\text{Area of } \Delta = \frac{1}{2} \cdot \frac{1}{2} \cdot 2 = \frac{1}{2}$$

14. The area bounded by the curver $4y^2 = x^2(4 - x)(x - 2)$ is equal to:

(1) $\frac{3\pi}{2}$

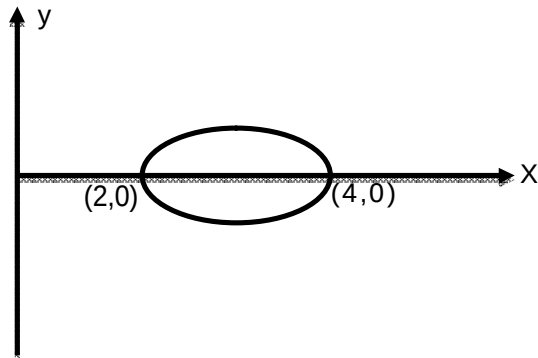
(2) $\frac{\pi}{16}$

(3) $\frac{\pi}{8}$

(4) $\frac{3\pi}{8}$

Ans. (1)

Sol. domain of $4y^2 = x^2(4-x)(x-2)$



$$\text{Area of loop} = 2 \times \frac{1}{2} \times \int_2^4 x \sqrt{(4-x)(x-2)} dx$$

Put $x = 4 \sin^2 \theta + 2 \cos^2 \theta$

$$dx = (8 \sin \theta \cos \theta - 4 \cos \theta \sin \theta) d\theta$$

$$= 4 \sin \theta \cos \theta d\theta$$

$$= \int_0^{\pi/2} (4 \sin^2 \theta + 2 \cos^2 \theta) \sqrt{(2 \cos^2 \theta)(2 \sin^2 \theta)} (4 \sin \theta \cos \theta) d\theta$$

$$= \int_0^{\pi/2} (4 \sin^2 \theta + 2 \cos^2 \theta) 8 (\cos \theta \sin \theta)^2 d\theta$$

$$= \int_0^{\pi/2} 32 \sin^4 \theta \cos^2 \theta d\theta + \int_0^{\pi/2} 16 \sin^2 \theta \cos^4 \theta d\theta$$

Using wallis theorem

$$= 32 \cdot \frac{3 \cdot 1 \cdot 1}{6 \cdot 4 \cdot 2} \frac{\pi}{2} + 16 \cdot \frac{3 \cdot 1 \cdot 1}{6 \cdot 4 \cdot 2} \frac{\pi}{2}$$

$$= \pi + \pi / 2 = 3\pi / 2$$

15. Define a relation R over a class of $n \times n$ real matrices A and B as “ARB iff there exists a non-singular matrix P such that $PAP^{-1} = B$ ”

The which of the following is true?

- (1) R is reflexive, symmetric but not transitive
- (2) R is symmetric, transitive but not reflexive,
- (3) R is an equivalence relation
- (4) R is reflexive, transitive but not symmetric

Ans. (3)

Sol. For reflexive

$$(B, B) \in R \Rightarrow B = PBP^{-1}$$

Which is true for $P = I$

$\therefore R$ is Reflexive

For symmetry

As $(B, A) \in R$ for matrix P

$$B = PAP^{-1} \Rightarrow P^{-1}B = P^{-1}PAP^{-1}$$

$$\Rightarrow P^{-1}BP = IAP^{-1}P = IAI$$

$$P^{-1}BP = A \Rightarrow A = P^{-1}BP$$

$$\therefore (A, B) \in R \text{ for matrix } P^{-1}$$

$\therefore R$ is symmetric

For transitivity

$$B = PAP^{-1} \text{ and } A = PCP^{-1}$$

$$\Rightarrow B = P(PCP^{-1})P^{-1}$$

$$\Rightarrow B = P^2C(P^{-1})^2 \Rightarrow B = P^2C(P^2)^{-1}$$

$$\therefore (B, C) \in R \text{ for matrix } P^2$$

$\therefore R$ is transitive

So R is equivalence

16. If P and Q are two statements, then which of the following compound statement is a tautology?

$$(1) ((P \Rightarrow Q) \wedge \sim Q) \Rightarrow P$$

$$(2) ((P \Rightarrow Q) \wedge \sim Q) \Rightarrow \sim P$$

$$(3) ((P \Rightarrow Q) \wedge \sim Q) \Rightarrow (P \wedge Q)$$

$$(4) ((P \Rightarrow Q) \wedge \sim Q) \Rightarrow Q$$

Ans. (2)

Sol. $(P \Rightarrow Q) \wedge \sim Q$

$$\equiv (\sim P \vee Q) \wedge \sim Q$$

$$\equiv (\sim P \vee \sim Q) \vee (Q \wedge \sim Q)$$

$$\equiv \sim (P \vee Q)$$

Now,

$$(1) \quad \sim (P \vee Q) \Rightarrow P$$

$$\equiv (P \vee Q) \vee P$$

$$\equiv P \vee Q$$

$$(2) \quad \sim (P \vee Q) \Rightarrow \sim P$$

$$\equiv (P \vee Q) \vee \sim P$$

$$\equiv T$$

$$(3) \quad \sim (P \vee Q) \Rightarrow (P \wedge Q)$$

$$\equiv (P \vee Q) \vee (P \wedge Q)$$

$$\equiv P \vee Q$$

$$(4) \quad \sim (P \vee Q) \Rightarrow Q$$

$$\equiv (P \vee Q) \vee Q$$

$$\equiv P \vee Q$$

- 17.** Consider a hyperbola $H : x^2 - 2y^2 = 4$. Let the tangent at a point $P(4, \sqrt{6})$ meet the x-axis at Q and latus rectum at $R(x_1, y_1)$, $x_1 > 0$. If F is a focus of H which is nearer to the point P, then the area of ΔQFR is equal to:

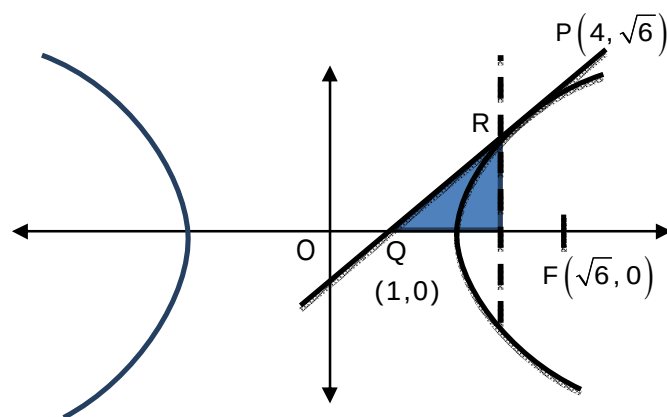
$$(1) \sqrt{6} - 1$$

$$(2) 4\sqrt{6} - 1$$

$$(3) 4\sqrt{6}$$

$$(4) \frac{7}{\sqrt{6}} - 2$$

Ans.
Sol.



Tangent at $P(4, \sqrt{6})$

$$4(x) - 2 \cdot \sqrt{6}(y) = 4$$

$$\Rightarrow 2x - \sqrt{6}(y) = 2 \quad \dots(1)$$

For Q, put $y = 0$

$$Q(1, 0)$$

Equation of Latus rectum:

$$x = ae = 2\sqrt{\frac{3}{2}} = \sqrt{6} \quad \dots(2)$$

Solving (1) & (2), we get

$$R\left(\sqrt{6}, 2 - \frac{2}{\sqrt{6}}\right)$$

$$\begin{aligned}
 \text{Area of } \triangle QFR &= \frac{1}{2} \times QF \times FR \\
 &= \frac{1}{2} (\sqrt{6} - 1) \left(2 - \frac{2}{\sqrt{6}} \right) \\
 &= \frac{7}{\sqrt{6}} - 2
 \end{aligned}$$

18. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined as

$$f(x) = \begin{cases} \frac{\sin(a+1)x + \sin 2x}{2x} & , \text{ If } x < 0 \\ b & , \text{ If } x = 0 \\ \frac{\sqrt{x + bx^3} - \sqrt{x}}{bx^{5/2}} & , \text{ If } x > 0 \end{cases}$$

If f is continuous at $x = 0$, then the value of $a + b$ is equal

(1) -2

(2) $-\frac{2}{5}$

(3) $-\frac{3}{2}$

(4) -3

Ans. (3)

Sol. ' f ' is continuous at $x = 0$

$$\Rightarrow f(0^-) = f(0) = f(0^+)$$

$$f(0^-) = \lim_{x \rightarrow 0^-} \frac{\sin(a+1)x + \sin 2x}{2x}$$

$$\lim_{x \rightarrow 0^-} \left\{ \frac{\sin(a+1)x}{(a+1)x} \cdot \frac{(a+1)}{2} + \frac{\sin(2x)}{2x} \right\}$$

$$= \frac{a+1}{2} + 1 \quad \dots(1)$$

$$f(0^+) = \lim_{x \rightarrow 0^+} \frac{\sqrt{x + bx^3} - \sqrt{x}}{bx^{5/2}}$$

$$= \lim_{x \rightarrow 0^+} \frac{bx^3}{b \cdot x^{5/2} \cdot (\sqrt{x + bx^3} + \sqrt{x})}$$

$$= \lim_{x \rightarrow 0^+} \frac{1}{\sqrt{1 + bx^2} + 1}$$

$$= \frac{1}{2} \quad \dots(2)$$

$$f(0) = b \quad \dots(3)$$

From (1), (2) and (3)

$$\therefore \frac{a+1}{2} + 1 = \frac{1}{2} = b$$

$$\Rightarrow a = -2 \text{ \& } b = \frac{1}{2}$$

$$\text{Thus, } a + b = -\frac{3}{2}$$

19. Let $y = y(x)$ be the solution of the differential equation $\frac{dy}{dx} = (y + 1) \left((y + 1)e^{\frac{x^2}{2}} - x \right)$, $0 < x < 2.1$, with $y(2) = 0$. Then the value of $\frac{dy}{dx}$ at $x = 1$ is equal to:

(1) $\frac{e^{\frac{5}{2}}}{(1 + e^2)^2}$

(2) $\frac{5e^{\frac{1}{2}}}{(e^2 + 1)^2}$

(3) $-\frac{2e^2}{(1 + e^2)^2}$

(4) $\frac{-e^{\frac{3}{2}}}{(e^2 + 1)^2}$

Ans. (4)

Sol.
$$\frac{dy}{dx} = (y + 1) \left((y + 1)e^{\frac{x^2}{2}} - x \right)$$

$$\Rightarrow \frac{-1}{(y + 1)^2} \frac{dy}{dx} - x \left(\frac{1}{y + 1} \right) = -e^{\frac{x^2}{2}}$$

Put, $\frac{1}{y + 1} = z$

$$-\frac{1}{(y + 1)^2} \cdot \frac{dy}{dx} = \frac{dz}{dx}$$

$$\therefore \frac{dz}{dx} + z(-x) = -e^{\frac{x^2}{2}}$$

$$\text{I.F} = e^{\int -x dx} = e^{-\frac{x^2}{2}}$$

$$z \cdot \left(e^{-\frac{x^2}{2}} \right) = - \int e^{-\frac{x^2}{2}} \cdot e^{\frac{x^2}{2}} dx = - \int 1 \cdot dx = -x + C$$

$$\Rightarrow \frac{e^{-\frac{x^2}{2}}}{y + 1} = -x + C \quad \dots(1)$$

Given $y = 0$ at $x = 2$

Put in (1)

$$\frac{e^{-2}}{0 + 1} = -2 + C$$

$$C = e^{-2} + 2 \quad \dots(2)$$

From (1) and (2)

$$y + 1 = \frac{e^{-x^2/2}}{e^{-2} + 2 - x}$$

Again, at $x = 1$

$$\Rightarrow y + 1 = \frac{e^{3/2}}{e^2 + 1}$$

$$\Rightarrow y + 1 = \frac{e^{3/2}}{e^2 + 1}$$

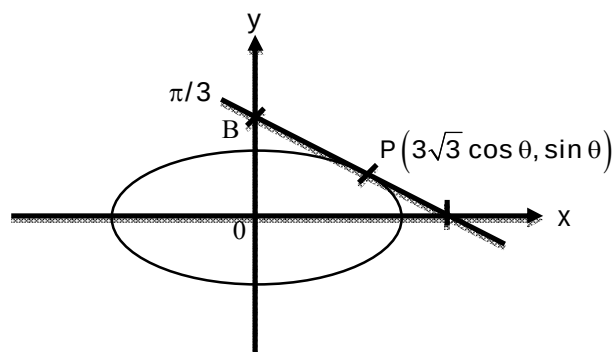
$$\therefore \left. \frac{dy}{dx} \right|_{x=1} = \frac{e^{3/2}}{e^2 + 1} \left(\frac{e^{3/2}}{e^2 + 1} \times e^{1/2} - 1 \right)$$

$$= -\frac{e^{3/2}}{(e^2 + 1)^2}$$

20. Let a tangent be drawn to the ellipse $\frac{x^2}{27} + y^2 = 1$ at $(3\sqrt{3} \cos \theta, \sin \theta)$ where $\theta \in \left(0, \frac{\pi}{2}\right)$. Then the value of θ such that the sum of intercepts on axes made by tangent is minimum is equal to :

- (1) $\frac{\pi}{8}$
- (2) $\frac{\pi}{6}$
- (3) $\frac{\pi}{3}$
- (4) $\frac{\pi}{4}$

Ans. (2)
Sol.



Equation of tangent

$$\frac{x}{3\sqrt{3}} \cos \theta + y \sin \theta = 1$$

$$A\left(\frac{3\sqrt{3}}{\cos \theta}, 0\right), B\left(0, \frac{1}{\sin \theta}\right)$$

$$\text{Now sum of intercept} = \frac{3\sqrt{3}}{\cos \theta} + \frac{1}{\sin \theta}$$

$$\text{Let } y = 3\sqrt{3} \sec \theta + \cos \theta$$

$$y' = 3\sqrt{3} \sec \theta \tan \theta - \sin \theta \cot \theta$$

$$y' = 0 \Rightarrow \tan \theta = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = \frac{\pi}{6}$$

SECTION – B

- 1.** Let P be a plane containing the line $\frac{x-1}{3} = \frac{y+6}{4} = \frac{z+5}{2}$ and parallel to the line $\frac{x-3}{4} = \frac{y-2}{-3} = \frac{z+5}{7}$. If the point $(1, -1, \alpha)$ lies on the plane P, then the value of $|5\alpha|$ is equal to _____.

Ans. (12)

Sol. DR's of normal $\vec{n} \equiv \vec{b}_1 \times \vec{b}_2$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 2 \\ 4 & -3 & 7 \end{vmatrix}$$

$$(34, -13, -25)$$

$$P \equiv 34(x-1) - 13(y+6) - 25(z+5) = 0$$

$Q(1, -1, \alpha)$ lies on P.

$$\Rightarrow 3(1-1) - 13(-1+6) - 25(\alpha+5) = 0$$

$$\Rightarrow -25(\alpha+5) = 65$$

$$\Rightarrow +5\alpha = -38$$

$$\Rightarrow |5\alpha| = 38$$

2.
$$\sum_{r=1}^{10} r! (r^3 + 6r^2 + 2r + 5) = \alpha (11!)$$

Then the value of α is equal to _____.

Ans. (160)

Sol. $T_r = r! ((r+1)(r+2)(r+3) - 9r - 1)$

$$= (r+3)! - 9r \cdot r! - r!$$

$$= (r+3)! - 9(r+1-1)r! - r!$$

$$= (r+3)! - 9(r+1)! + 8r!$$

$$= \{(r+3)! - (r+1)!\} - 8\{(r+1)! - r!\}$$

Now,
$$\sum_{r=1}^{10} T_r = \{13! - 12! - 3! - 2!\} - 8\{11! - 1!\}$$

$$\begin{aligned}
&= 13! + 12! - 811! \\
&= (13 \times 12 + 12 - 8)11! \\
&= 160 \times 11! \\
\text{Thus, } \alpha &= 160
\end{aligned}$$

3. The term independent of x in the expansion of $\left[\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}} \right]^{10}$, $x \neq 1$, is equal to _____.

Ans. (210)

Sol. Given, $\left((x^{1/3} + 1) - \left(\frac{\sqrt{x} + 1}{\sqrt{x}} \right) \right)^{10} = (x^{1/3} - x^{-1/2})^{10}$

General term, $T_{r+1} = {}^{10}C_r (x^{1/3})^{10-r} (-x^{-1/2})^r$

For term independent of x

$$\begin{aligned}
\frac{10-r}{3} - \frac{r}{2} &= 0 \Rightarrow 20 - 2r - 3r = 0 \\
\Rightarrow r &= 4
\end{aligned}$$

Therefore required term, $T_5 = {}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$

4. Let nC_r denote the binomial coefficient of x^r in the expansion of $(1+x)^n$. If $\sum_{k=0}^{10} (2^2 + 3k) {}^nC_k = \alpha \cdot 3^{10} + \beta \cdot 2^{10}$, $\alpha, \beta \in \mathbb{R}$, then $\alpha + \beta$ is equal to _____.

Bonus

Sol. n must be equal to 10

$$\begin{aligned}
&\sum_{k=0}^{10} (2^2 + 3k) {}^nC_k \\
&= \sum_{k=0}^{10} (4 + 3k) {}^nC_k \\
&= 4 \sum_{k=0}^{10} {}^nC_k + 3 \sum_{k=0}^{10} k {}^nC_k \\
&= 4(2^{10}) + 3 \times 10 \times 2^9 \\
&= 19 \times 2^{10} \\
\therefore \alpha &= 0 \text{ and } \beta = 19 \\
\text{Thus, } \alpha + \beta &= 19
\end{aligned}$$

5. Let $P(x)$ be a real polynomial of degree 3 which vanishes at $x = -3$. Let $P(x)$ have local minima at $x = 1$, local maxima at $x = -1$ and $\int_{-1}^1 P(x) dx = 18$, then the sum of all the coefficients of the polynomial $P(x)$ is equal to _____.

Ans. (8)

Sol. $P'(x) = a(x+1)(x-1)$

$$\therefore P(x) = \frac{ax^3}{3} - ax + C$$

$$P(-3) = 0 \text{ (given)}$$

$$\Rightarrow a(-9+3) + C = 0$$

$$\Rightarrow 6a = C \quad \dots(i)$$

$$\text{Also, } \int_{-1}^1 P(x) dx = 18 \Rightarrow \int_{-1}^1 \left(a \left(\frac{x^3}{3} - x \right) + C \right) dx = 18$$

$$\Rightarrow 0 + 2C = 18 \Rightarrow C = 9$$

from (i)

$$a = \frac{3}{2}$$

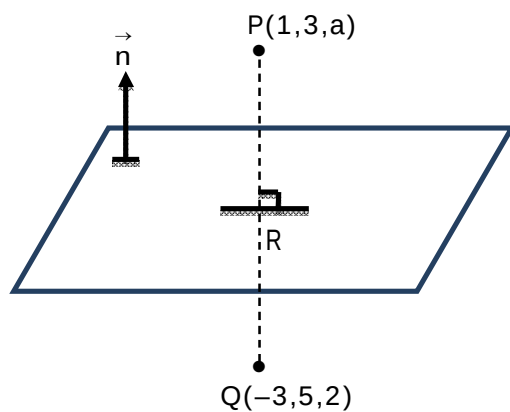
$$\therefore P(x) = \frac{x^3}{2} - \frac{3}{2}x + 9$$

$$\text{Sum of co-efficient} = -1 + 9 = 8$$

6. Let the mirror image of the point $(1, 3, a)$ with respect to the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) - b = 0$ be $(-3, 5, 2)$. Then, the value of $|a+b|$ is equal to _____.

Ans. (1)

Sol.



$$\text{Plane : } 2x - y + z = b$$

$$R \equiv \left(-1, 4, \frac{a+2}{2} \right) \rightarrow \text{on plane}$$

$$\therefore -2 - 4 + \frac{a+2}{2} = b$$

$$\Rightarrow a + 2 = 2b + 12 \Rightarrow a = 2b + 10 \quad \dots(1)$$

$$PQ < 4, -2, a-2 >$$

$$\therefore \frac{2}{4} = \frac{-1}{-2} = \frac{1}{a-2} \Rightarrow a-2 = 2 \Rightarrow a = 4, b = -3$$

$$\therefore |a+b| = 1$$

7. If $f(x)$ and $g(x)$ are two polynomials such that the polynomial $P(x) = f(x^3) + x g(x^3)$ is divisible by $x^2 + x + 1$, then $P(1)$ is equal to _____.

Ans. 0

Sol. roots of $x^2 + x + 1$ are ω and ω^2 now

$$Q(\omega) = f(1) + \omega g(1) = 0 \quad \dots(1)$$

$$Q(\omega^2) = f(1) + \omega^2 g(1) = 0 \quad \dots(2)$$

Adding (1) and (2)

$$\Rightarrow 2f(1) - g(1) = 0$$

$$\Rightarrow g(1) = 2f(1)$$

$$\Rightarrow f(1) = g(1) = 0$$

Therefore, $Q(1) = f(1) + g(1) = 0 + 0 = 0$

8. Let I be an identity matrix of order 2×2 and $P = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}$. Then the value of $n \in \mathbb{N}$ for which $P^n = 5I - 8P$ is equal to _____.

Ans. (6)

$$\text{Sol. } P^2 = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -5 & 4 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} -1 & 1 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -15 & 11 \end{bmatrix}$$

$$P^6 = \begin{bmatrix} -4 & 3 \\ -15 & 11 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} -11 & 8 \\ -40 & 29 \end{bmatrix}$$

$$\text{and } 5I - 8P = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 8 \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -11 & 8 \\ -40 & 29 \end{bmatrix}$$

$$\Rightarrow P^6 = 5I - 8P$$

Thus, $n = 6$

9. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfy the equation $f(x+y) = f(x) \cdot f(y)$ for all $x, y \in \mathbb{R}$ and $f(x) \neq 0$ for any $x \in \mathbb{R}$. If the function f is differentiable at $x = 0$ and $f'(0) = 3$, then $\lim_{h \rightarrow 0} \frac{1}{h} (f(h) - 1)$ is equal to _____.

Ans. (3)

Sol. $f(x+y) = f(x) \cdot f(y)$ then

$$\Rightarrow f(x) = a^x$$

$$\Rightarrow f'(x) = a^x \ell n a$$

$$\Rightarrow f'(0) = \ell n a = 3 \text{ (given } f'(0) = 3)$$

$$\Rightarrow a = e^3$$

$$\therefore f(x) = (e^3)^x = e^{3x}$$

$$\text{Now, } \lim_{h \rightarrow 0} \frac{f(h) - 1}{h} = \lim_{h \rightarrow 0} \left(\frac{e^{3h} - 1}{3h} \times 3 \right) = 1 \times 3 = 3$$

10. Let $y = y(x)$ be the solution of the differential equation $x dy - y dx = \sqrt{x^2 - y^2} dx$, $x \geq 1$, with $y(1) = 0$. If the area bounded by the line $x = 1$, $x = e^\pi$, $y = 0$ and $y = y(x)$ is $\alpha e^{2\pi} + \beta$, then the value of $10(\alpha + \beta)$ is equal to

Ans. (4)

$$\text{Sol. } x dy - y dx = \sqrt{x^2 - y^2} dx \Rightarrow \frac{x dy - y dx}{x^2} = \frac{1}{x} \sqrt{1 - \frac{y^2}{x^2}} dx \Rightarrow \int \frac{d\left(\frac{x}{y}\right)}{\sqrt{1 - \left(\frac{y}{x}\right)^2}} = \int \frac{dx}{x}$$

$$\Rightarrow \sin^{-1}\left(\frac{y}{x}\right) = \ell n|x| + c$$

$$\text{At } x = 1, y = 0 \Rightarrow c = 0$$

$$y = x \sin(\ell n x)$$

$$A = \int_1^{e^\pi} x \sin(\ell n x) dx$$

$$x = e^t, dx = e^t dt = \int_0^\pi e^{2t} \sin(t) dt$$

$$\alpha e^{2\pi} + \beta = \left(\frac{e^{2t}}{5} (2 \sin t - \cos t) \right)_0^\pi = \frac{1 + e^{2\pi}}{5}$$

$$\alpha = \frac{1}{5}, \beta = \frac{1}{5}$$

$$\text{Thus, } 10(\alpha + \beta) = 4$$