

JEE-Main-25-07-2022-Shift-2 (Memory Based)

MATHEMATICS

Question: The number of bijective function $f(1,3,5,7,\dots,99) \rightarrow (2,4,6,8,\dots,100)$ if $f(3) \geq f(5) \geq \dots \geq f(99)$ is:

Options:

(a) ${}^{50}C_1$

(b) ${}^{50}C_2$

(c) $\frac{50!}{2}$

(d) ${}^{50}C_3 \times 3!$

Answer: (a)

Solution:

Bijective function means one-one and onto.

That means for every input unique output which is non-repeating so, set $A(1,3,5,7,\dots,99)$

has 50 elements and set $B(2,4,6,\dots,100)$ has 50 elements.

Such that $f(3) \geq f(5) \geq \dots \geq f(99)$

This can be done in ${}^{50}C_1$ ways

Question: If $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{5}$ and $P(A \cup B) = \frac{1}{2}$ then $P\left(\frac{A}{B'}\right) + P\left(\frac{A'}{B}\right) =$

Options:

(a) $\frac{5}{8}$

(b) $\frac{4}{9}$

(c) $\frac{29}{24}$

(d) 3

Answer: (c)

Solution:

$$\text{As } P(A) = \frac{1}{3}, P(B) = \frac{1}{5} \text{ and } P(A \cup B) = \frac{1}{2}$$

$$\text{So, } P(A \cap B) = \frac{1}{30}$$

$$\text{Now, } P\left(\frac{A}{B'}\right) = \frac{P(A \cap B')}{P(B')} = \frac{\frac{1}{3} - \frac{1}{30}}{\frac{4}{5}} = \frac{3}{8}$$

$$\text{And } P\left(\frac{A'}{B}\right) = \frac{P(A' \cap B)}{P(B)} = \frac{\frac{1}{5} - \frac{1}{30}}{\frac{1}{6}} = \frac{5}{6}$$

$$\text{So, } P\left(\frac{A}{B'}\right) + P\left(\frac{A'}{B}\right) = \frac{3}{8} + \frac{5}{6} = \frac{29}{24}$$

Question: Let $f(x) = [x^2 - 2x] + |5x - 7|$, and let m be minimum value of $f(x)$ and M be maximum value of $f(x)$ in $\left[\frac{5}{4}, 2\right]$, then:

Options:

(a) $m = -1, M = 2$

(b) $m = 0, M = 3$

(c) $m = -1, M = 4$

(d) $m = -2, M = 2$

Answer: (a)

Solution:

$$\text{For } x \in \left[\frac{5}{4}, 2\right], [x^2 - 2x] = -1$$

So, $f(x) = -1 + |5x - 7|$ is least at $x = \frac{7}{5}$ and greatest at $x = 2$

$$m = f\left(\frac{7}{5}\right) = -1$$

$$\text{And } M = f(2) = 2$$

Question: The tangent at the point at $A(1, 3)$ & $B(1, -1)$ on the parabola $y^2 - 2x - 2y = 1$ meet at point P. Find area of $\triangle PAB$.

Options:

(a) 4

(b) 6

(c) 7

(d) 8

Answer: (b)

Solution:

Tangent at $A(1, 3)$

$$3y - (x+1) - (y+3) = 1$$

$$\Rightarrow x - 2y + 5 = 0$$

Tangent at $B(1, -1)$

$$-y - (x+1) - (y-1) = 1$$

$$x + 2y - 1 = 0$$

$$\therefore P \text{ is } \left(-2, \frac{3}{2}\right)$$

$$\therefore \text{Area of } \triangle PAB = \frac{1}{2} \begin{vmatrix} 1 & 3 & 1 \\ 1 & -1 & 1 \\ -2 & \frac{3}{2} & 1 \end{vmatrix}$$

$$= \frac{1}{2} \left[1 - \left(-1 - \frac{3}{2}\right) - 3(1+2) + 1\left(\frac{3}{2} - 2\right) \right]$$

$$= \frac{1}{2} \left[-\frac{5}{2} - 9 - \frac{1}{2} \right]$$

$$= 6$$

Question: If $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$ & $\vec{a} \times \vec{b} = 2\hat{i} - \hat{k}$, $\vec{a} \cdot \vec{b} = 3$, find projection of $\vec{b}$ on $\vec{a} - \vec{b}$

Options:

(a) $\frac{2}{\sqrt{21}}$

(b) $\frac{\sqrt{3}}{7}$

(c) $\frac{\sqrt{7}}{3}$

(d) $\frac{2}{3}$

Answer: (a)

Solution:

$$\text{As } |\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$$

$$(\sqrt{5})^2 + (3)^2 = (\sqrt{6})^2 |\vec{b}|^2$$

$$\text{Now, projection of } \vec{b} \text{ on } \vec{a} - \vec{b} = \frac{(\vec{b}) \cdot (\vec{a} - \vec{b})}{|\vec{a} - \vec{b}|}$$

$$= \frac{\vec{a} \cdot \vec{b} - |\vec{b}|^2}{|\vec{a} - \vec{b}|}$$

$$\text{Now, } |\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b} = \frac{7}{3}$$

$$|\vec{a} - \vec{b}| = \frac{\sqrt{7}}{3}$$

$$\therefore \text{Projection is } \frac{3 - \frac{7}{3}}{\frac{\sqrt{7}}{3}} = \frac{2}{\sqrt{21}}$$

Question: Shortest distance between the lines $\frac{x+7}{-6} = \frac{y-6}{7} = 7$ and $\frac{7-x}{2} = y-2 = z-6$ is

Options:

(a) $2\sqrt{29}$

(b) 1

(c) $\frac{\sqrt{37}}{29}$

(d) $\frac{\sqrt{29}}{22}$

Answer: (a)

Solution:

$$L_1: \frac{x+7}{-6} = \frac{y-6}{7} = \frac{z-0}{1}$$

$$L_2: \frac{x-7}{-2} = \frac{y-2}{1} = \frac{z-6}{1}$$

$$d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$$

$$\text{Here, } \vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6 & 7 & 1 \\ -2 & 1 & 1 \end{vmatrix} = 6\hat{i} + 4\hat{j} + 8\hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = 14\hat{i} - 4\hat{j} + 6\hat{k}$$

$$\therefore d = \frac{|(14\hat{i} - 4\hat{j} + 6\hat{k}) \cdot (6\hat{i} + 4\hat{j} + 8\hat{k})|}{\sqrt{36 + 16 + 64}} = 2\sqrt{29}$$

Question: $\sin\left(\frac{\pi}{22}\right)\sin\left(\frac{3\pi}{22}\right)\sin\left(\frac{5\pi}{22}\right)\sin\left(\frac{7\pi}{22}\right)\sin\left(\frac{9\pi}{22}\right) = ?$

Answer: $\frac{1}{32}$

Solution:

$$\begin{aligned}
& \sin\left(\frac{\pi}{22}\right)\sin\left(\frac{3\pi}{22}\right)\sin\left(\frac{5\pi}{22}\right)\sin\left(\frac{7\pi}{22}\right)\sin\left(\frac{9\pi}{22}\right) \\
& \cos\left(\frac{\pi}{2}-\frac{\pi}{22}\right)\cos\left(\frac{\pi}{2}-\frac{3\pi}{22}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{22}\right)\cos\left(\frac{\pi}{2}-\frac{7\pi}{22}\right)\cos\left(\frac{\pi}{2}-\frac{9\pi}{22}\right) \\
& \cos\left(\frac{10\pi}{22}\right)\cos\left(\frac{8\pi}{22}\right)\cos\left(\frac{6\pi}{22}\right)\cos\left(\frac{4\pi}{22}\right)\cos\left(\frac{2\pi}{22}\right) \\
& \cos\left(\frac{\pi}{11}\right)\cos\left(\frac{2\pi}{11}\right)\cos\left(\frac{3\pi}{11}\right)\cos\left(\frac{4\pi}{11}\right)\cos\left(\frac{5\pi}{11}\right) = \frac{1}{2^5} = \frac{1}{32}
\end{aligned}$$

Question: $\sum_{n=1}^{21} \frac{3}{(4n-3)(4n+1)} = ?$

Answer: $\frac{63}{85}$

Solution:

$$T_n = \frac{3}{(4n-3)(4n+1)}$$

$$T_n = \frac{3}{4} \left(\frac{4}{(4n-3)(4n+1)} \right)$$

$$= \frac{3}{4} \left[\frac{(4n+1) - (4n-3)}{(4n-3)(4n+1)} \right]$$

$$T_n = \frac{3}{4} \left[\frac{1}{4n-3} - \frac{1}{4n+1} \right]$$

$$\Rightarrow T_1 = \frac{3}{4} \left(\frac{1}{1} - \frac{1}{5} \right)$$

$$\Rightarrow T_2 = \frac{3}{4} \left(\frac{1}{5} - \frac{1}{9} \right)$$

$\vdots$

$$T_{21} = \frac{3}{4} \left(\frac{1}{81} - \frac{1}{85} \right)$$

$$S_{21} = \frac{3}{4} \left(1 - \frac{1}{85} \right)$$

$$= \frac{3}{4} \left(\frac{84}{85} \right) = \frac{63}{85}$$

Question: Find remainder when $(11)^{1011} + (1011)^{11}$ is divided by 9.

Answer: 8.00

Solution:

$$\text{Given, } (11)^{1011} + (1011)^{11}$$

$$\Rightarrow (9+2)^{1011} + (1008+3)^{11}$$

$$\Rightarrow 9 \text{ Integer} + 2^{1011} + 9 \text{ Integer} + 311$$

$$\Rightarrow (2^3)^{337} + 3(3^2)^5$$

$$\Rightarrow (9-1)^{337}$$

$$\Rightarrow 9 \text{ Integer} - 1$$

$$\Rightarrow 9 \text{ Integer} - 1 - 8 + 8$$

$\therefore$ Remainder will be 8.

Question: $\lim_{x \rightarrow \frac{\pi}{4}} \frac{2\sqrt{2} - (\cos x + \sin x)^7}{\sqrt{2} - \sqrt{2} \sin 2x}$

Answer: 14.00

Solution:

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{2\sqrt{2} - (\cos x + \sin x)^7}{\sqrt{2} - \sqrt{2} \sin 2x}$$

Since its $\frac{0}{0}$ form, let's apply L-Hospital rule

$$\lim_{x \rightarrow \frac{\pi}{4}} 0 - \frac{7(\cos x + \sin x)^6 (-\sin x + \cos x)}{0 - \sqrt{2} \cos 2x \cdot (2)}$$

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{7(\cos x + \sin x)^5 (\cos^2 x - \sin^2 x)}{2\sqrt{2} \cos^2 x}$$

$$\frac{7}{2\sqrt{2}} (\sqrt{2})^5 = \frac{7(2 \times 2\sqrt{2})}{2\sqrt{2}} = 14$$

Question: If $x^2 + px^2 + qx + 1 = 0$ ($p < q$) has only one root α , then α belongs to:

Answer:

Solution:

$$f(0) = 1$$

$$\& f(-1) = -1 + p - q + 1 = p - q < 0$$

$$\therefore f(0) > 0 \& f(-1) < 0$$

$\therefore f(x)$ must have root between $(-1, 0)$