

JEE-Main-27-07-2022-Shift-2 (Memory Based)

MATHEMATICS

Question: Let $A = \begin{bmatrix} 4 & -2 \\ \alpha & \beta \end{bmatrix}$. If $A^2 + \gamma A + 18I = 0$, then $\det(A)$ equals:

Options:

(a) -18

(b) 18

(c) -50

(d) 50

Answer: (b)

Solution:

Characteristic equation of matrix:

$$\begin{bmatrix} 4 - \lambda & -2 \\ \alpha & \beta - \lambda \end{bmatrix} = 0$$

$$\Rightarrow 4\beta + \lambda^2 - (\beta + 4)\lambda + 2\alpha = 0$$

$$\therefore A^2 - (\beta + 4)A + 2\alpha I = 0$$

$$\Rightarrow \gamma = 0 - \beta + 4 \text{ \& } 2\alpha + 4\beta = 18$$

$$\det(A) = 4\beta + 2\alpha = 18$$

Question: The area of region enclosed by $y \leq 4x^2$, $x^2 \leq 9y$, $y \leq 4$ is equal to:

Options:

(a) $\frac{40}{3}$

(b) $\frac{56}{3}$

(c) $\frac{112}{3}$

(d) $\frac{80}{3}$

Answer: (d)

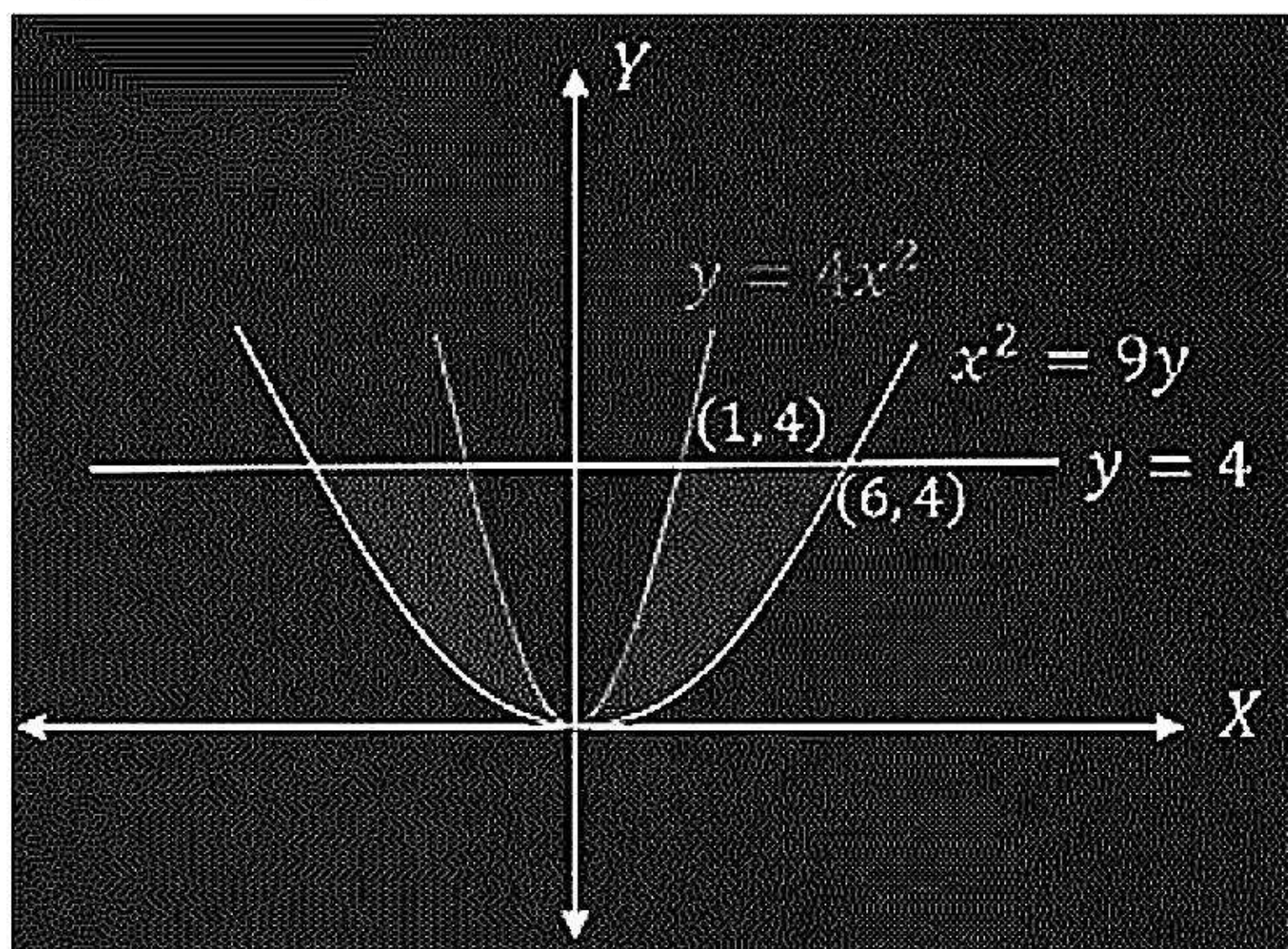
Solution:

$$\text{Required Area} = 2 \int_0^4 \left(3\sqrt{y} - \frac{\sqrt{y}}{2} \right) dy$$

$$= 2 \cdot \frac{5}{2} \int_0^4 \sqrt{y} dy$$

$$= 5 \left[\frac{2}{3} y^{\frac{3}{2}} \right]_0^4$$

$$= \frac{10}{3} (4)^2 = \frac{80}{3}$$



Question: If the length of the latus rectum of a parabola whose focus is (a, a) and tangent at its vertex is $x + y = a$, is 16. Then $|a|$ is equal to:

Options:

- (a) $2\sqrt{3}$
- (b) $2\sqrt{2}$
- (c) $4\sqrt{2}$
- (d) 4

Answer: (c)

Solution:

Length of perpendicular from focus to tangent at vertex:

$$l = \left| \frac{a}{\sqrt{2}} \right|$$

So length of latus rectum will be, $4l = 16$

$$\Rightarrow 2\sqrt{2}|a| = 16$$

$$\Rightarrow |a| = 4\sqrt{2}$$

Question: Let $f(x) = \frac{\left(729p(1+x)^{\frac{1}{7}}\right) - 3}{\left(729(1+qx)^{\frac{1}{3}}\right) - 9}$, and $f(x)$ is continuous at $x = 0$, then:

Options:

- (a) $21qf(0) - p = 0$

$$(b) \ 21q^2 f(0) - p^3 = 0$$

$$(c) \ 21p^2 f(0) - q^3 = 0$$

$$(d) \ p^2 f(0) - 7q^2 = 0$$

Answer: (a)

Solution:

$\lim_{x \rightarrow 0} f(x)$ exists if numerator of $f(x)$ is zero at $x = 0$.

Clearly, $p = 3$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{3 \left[(x+1)^{\frac{1}{7}} - 1 \right]}{9 \left[(1+qx)^{\frac{1}{3}} - 1 \right]}$$

$$= \frac{1}{3} \left(\frac{\frac{1}{7}}{\frac{q}{3}} \right) = \frac{1}{7q} = f(0)$$

$$\text{So, } 21qf(0) = 3 = p$$

$$\Rightarrow 21qf(0) - p = 0$$

Question: Let $f(x) = \min \{[x], [x-1], [x-2], \dots, [x-10]\}$ where $[\]$ denotes greatest integer

function. Then $\int_0^{10} (f(x) + |f(x)| + f^2(x)) dx$ is equal to:

Options:

(a) 55

(b) 385

(c) 5050

(d) 270

Answer: (b)

Solution:

$$\text{Clearly } f(x) = [x-10]$$

$$\text{Here } f(x) \leq 0 \ \forall \ x \in (0, 10)$$

$$\text{So, } \int_0^{10} (f(x) + |f(x)|) dx = 0$$

$$\text{Now, } \int_0^{10} f^2(x) dx = \int_0^{10} ([x]-10)^2 dx$$

$$= \int_0^1 100 dx + \int_1^2 81 dx + \int_2^3 64 dx + \dots + \int_9^{10} 1 dx$$

$$\begin{aligned}
 &= (1^2 + 2^2 + 3^2 + \dots + 10^2) \\
 &= \frac{10 \times 11 \times 21}{6} \\
 &= 385
 \end{aligned}$$

Question: The value of $\int_0^2 \left(|2x^3 - 3x| + \left[x - \frac{1}{2} \right] \right) dx$, where $[.]$ is greatest integer function is:

Options:

- (a) $\frac{7}{6}$
- (b) $\frac{19}{12}$
- (c) $\frac{17}{4}$
- (d) $\frac{3}{2}$

Answer: (c)

Solution:

$$\begin{aligned}
 &\text{Given, } \int_0^2 \left(|2x^3 - 3x| + \left[x - \frac{1}{2} \right] \right) dx \\
 &= \int_0^{\frac{\sqrt{3}}{2}} (3x - 2x^3) dx + \int_{\frac{\sqrt{3}}{2}}^2 (2x^3 - 3x) dx + \int_0^{\frac{1}{2}} \left[x - \frac{1}{2} \right] dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \left[x - \frac{1}{2} \right] dx + \int_{\frac{3}{2}}^2 \left[x - \frac{1}{2} \right] dx \\
 &= \left[\frac{3x^2 - x^4}{2} \right]_0^{\frac{\sqrt{3}}{2}} + \left[\frac{x^4 - 3x^2}{2} \right]_{\frac{\sqrt{3}}{2}}^2 + \left(-\frac{1}{2} \right) + 0 + \left(\frac{1}{2} \right) \\
 &= \frac{9}{8} + 2 + \frac{9}{8} \\
 &= \frac{17}{4}
 \end{aligned}$$

Question: If the line of intersection of the planes $ax + by = 3$ and $ax + by + cz = 0$ makes an angle 30° with the plane $y - z + 2 = 0$, then the direction cosines of line are:

Options:

- (a) $\frac{1}{\sqrt{2}}, 0, \frac{1}{2}$
- (b) $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0$

(c) $\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0$

(d) $\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0$

Answer: (b)

Solution:

Direction ratios of line of intersection $(b, -a, 0)$

As angle between this line and $y - z + 2 = 0$ is 30°

$$\therefore \sin \theta = \left| \frac{a}{\sqrt{a^2 + b^2} \cdot \sqrt{2}} \right|$$

$$\Rightarrow a^2 = b^2$$

$$\therefore \text{Possible combination is } \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0$$

Question: If $A = \begin{bmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta + \gamma & \alpha + \gamma & \alpha + \beta \end{bmatrix}$ and $\frac{\left| \text{adj} \left(\text{adj} \left(\text{adj} \left(\text{adj} (A) \right) \right) \right) \right|}{(\alpha - \beta)^{16} (\beta - \gamma)^{16} (\gamma - \alpha)^{16}} = 2^{32} \cdot 3^{16}$, where

α, β, γ are distinct natural number, then number of triplets of (α, β, γ) is _____.

Answer: 55.00

Solution:

$$A = \begin{bmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta + \gamma & \alpha + \gamma & \alpha + \beta \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_1$$

$$\Rightarrow |A| = (\alpha + \beta + \gamma) \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (\alpha + \beta + \gamma)(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)$$

$$\therefore \left| \text{adj} \left(\text{adj} \left(\text{adj} \left(\text{adj} (A) \right) \right) \right) \right| = |A|^{(2)^4} = |A|^{16}$$

$$|A| = (\alpha + \beta + \gamma)(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)$$

$$\text{Clearly } (\alpha + \beta + \gamma)^{16} = 2^{32} \cdot 3^{16}$$

$$\Rightarrow (\alpha + \beta + \gamma) = 12$$

$$\text{Number of positive integral solutions} = {}^{11}C_2 = 55$$

Question: $\frac{(2^3 - 1^3)}{(1 \times 7)} + \frac{\{(4^3 - 3^3) + (2^3 - 1^3)\}}{(2 \times 11)} + \frac{\{(6^3 - 5^3) + (4^3 - 3^3) + (2^3 - 1^3)\}}{(3 \times 15)} + \dots$ upto 15

terms

Answer: 120.00

Solution:

$$\frac{2^3 - 1^3}{1 \times 7} + \frac{4^3 - 3^3 + 2^3 - 1^3}{2 \times 11} + \dots$$

$$= 1 + 2 + 3 + \dots$$

$$= \left(\frac{15 \times 16}{2} \right)$$

$$= 120$$

Question: Domain of $f(x) = \sin^{-1}[2x^2 - 3] + \log_2 \left(\log_{\frac{1}{2}}(x^2 - 5x + 5) \right)$

Answer: $1, \frac{5 - \sqrt{5}}{2}$

Solution:

$$-1 \leq [2x^2 - 3] \leq 1$$

$$-1 \leq (2x^2 - 3) < 2$$

$$2 \leq 2x^2 < 5$$

$$1 \leq x^2 < \frac{5}{2} \quad \dots (1)$$

$$\log_{\frac{1}{2}}(x^2 - 5x + 5) > 0$$

$$0 < x^2 - 5x + 5 < 1$$

$$\Rightarrow x^2 - 5x + 5 = 0 \text{ and } x^2 - 5x + 4 < 0$$

$$x \in \left(-\infty, \frac{5 - \sqrt{5}}{2} \right) \cup \left(\frac{5 + \sqrt{5}}{2}, \infty \right) \quad \dots (2)$$

and $x \in (1, 4)$

Taking intersection of (1) and (2)

$$x \in \left(1, \frac{5 - \sqrt{5}}{2} \right)$$

Question: Let n^{th} term of any sequence is given by $T_n = \frac{-1^3 + 2^3 - 3^3 + 4^3 + \dots + (2n)^3}{n(4n + 3)}$, then

$\sum_{n=1}^{15} T_n$ is equal to ____.

Answer: 120.00

Solution:

$$T_n = \frac{2[2^3 + 4^3 + \dots + (2n)^3] - [1^3 + 2^3 + 3^3 + \dots + (2n)^3]}{n(4n+3)}$$

$$T_n = \frac{16\left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{2n(2n+1)}{2}\right)^2}{n(4n+3)}$$

$$= \frac{n^2(4n+3)}{n(4n+3)} = n$$

$$\therefore \sum_{n=1}^{15} T_n = \frac{15 \times 16}{2} = 120$$